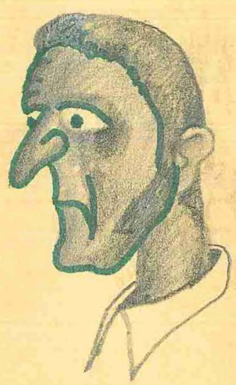
Probability & Statistics

R.J. Marks II Class Notes

Rose-Hulman Institute of Technology (1970)

PROBABILTY



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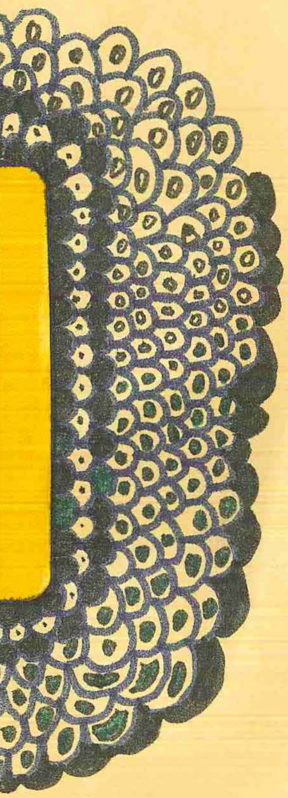
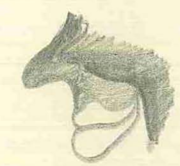




PR08

PROB

9-10-70

MATHEMATICAL MODEL APROXIMATES REALITY

9-11-70

1) LET A & B BE SETS

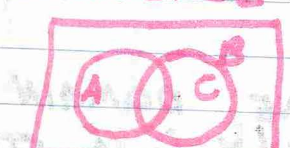
2) DEFINITION: $x \in A \Rightarrow x$ IS AN ELEMENT OF SET A

3) DEFINITION: $A \subset B \Rightarrow$ A IS A SUBSET OF B (IFF EVERY ELEMENT IN A IS ALSO IN B)

4) DEFINITION: A = B IFF ~~____~~ $A \subset B$ AND $B \subset A$

5) DEFINITION: $A \cup B \Rightarrow$ A UNION B
$A \cup B$ IS ~~TE~~ THE SET OF ALL ELEMENTS WHICH ARE EITHER IN A OR B OR BOTH

6) VENN DIAGRAM
TAKE UNIVERSE



7) DEFINITION: $A \cap B \Rightarrow$ A INTERSECT B
$A \cap B$ IS THE SET OF ALL ELEMENTS WHICH ARE IN BOTH A & B

8) DEFINITION: ϕ; THE EMPTY SET

9) DEFINITION: A & B DISJOINT IFF $A \cap B = \phi$

10) DEFINITION: RELATIVE TO A UNIVERSAL SET U, THE COMPLEMENT OF A

(CONT)

PROB

DENOTED BY $\bar{A}$, IS THE SET OF ALL ELEMENTS IN U NOT IN A

11) DEFINITION: A X B REPRESENTS THE CARTESIAN PRODUCT OF A & B.

$A \times B = \{(X_1, X_2) \mid X_1 \in A \text{ \& } X_2 \in B\}$

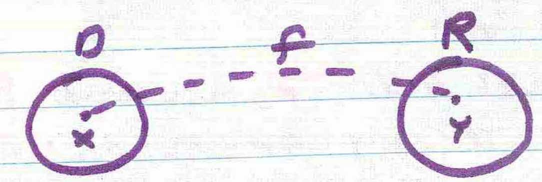
EXAMPLE:

A = {1,2,3}   B = {4,5}

A X B = {(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)}

9-14-70

FUNCTION - DOMAIN & RANGE



$Y = f(x)$

MAY RESTRICT DOMAIN

SHOULD DEFINE DOMAIN

D = {X | X ~~is~~ A STUDENT IN THIS ROOM}

X	Y
MARKS	1
KATO	2
SIMPSON	3
BALOGH	4
⋮	⋮

DEFINITION: AN ELEMENT FUNCTION f DEFINED ON A SET S IS A FUNCTION WHOSE DOMAIN IS S. $f(\omega)$ IS THE VALUE OF f FOR $\omega \in S$

DEFINITION: A CLASS IS A SET, EACH OF WHOSE ELEMENTS ARE SETS

EX) $S = \{1,2,3\}$

FORM CLASS OF ALL SUBSETS

$\tilde{\mathcal{F}} = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

PERFORM FUNCTION ON $\tilde{\mathcal{F}}$

$n(x)$ = # OF ELEMENTS IN SET

X	Y
ϕ	0
{1}	1
{2}	1
{3}	1
{1,2}	2
{1,3}	2
{2,3}	2
{1,2,3}	3

DEFINITION: A REAL VALUED FUNCTION f WHOSE DOMAIN IS A CLASS $\tilde{\mathcal{F}}$ (WHICH IS CLOSED UNDER $\cup$ & $\cap$) IS CALLED A SET FUNCTION ON $\tilde{\mathcal{F}}$. $f(A)$ IS THE

(CONT.)

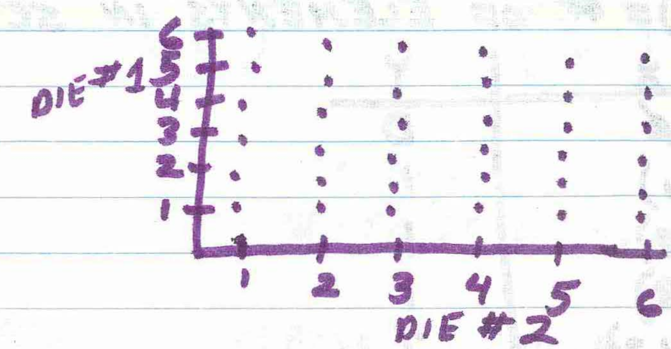
P (A) IS THE PROBABILITY OF A

VALUE OF FOR $A \in \tilde{\mathcal{F}}$

1) DEFN.: AN EXPERIMENT IS ANY OPERATION WHOSE OUTCOME CANNOT BE PREDICTED WITH CERTAINTY

2) DEFN: THE SAMPLE SPACE S FOR AN EXPERIMENT IS THE SET OF ALL POSSIBLE OUTCOMES THE EXPERIMENTS

EX. TOSSING DICE



9-15-70

AN EVENT IS A SUBSET OF A SAMPLE SPACE. AN EVENT OCCURS IF ANY ONE OF ITS ELEMENTS IS AN OUTCOME OF THE EXPERIMENT

LET A BE THE EVENT THAT A TOTAL OF 7 ~~APPEARANCES~~ APPEARS (2 DICE)

LET B BE THE EVENT THAT A 4 APEARS ON DIE #1

$A \cup B$ IS THE EVENT $\ni$ EITHER A TOTAL OF 7 APPEARS OR A 4 APPEARS ON DIE #1

$A \cap B$ IS THE EVENT THAT A TOTAL OF 7 APEARS AND A 4 APPEARS ON DIE #1

A PROBABILITY MEASURE IS ~~A~~ <u>SET</u> <u>FUNCTION</u>

<u>DEFINITION:</u> A PROBABILITY MEASURE P IS A REAL VALUED SET FUNC. DEFINED AS THE CLASS OF ALL EVENTS IN A SAMPLE SPACE S $\ni$ THE FOLLOWING AXIOMS HOLD

i) $P(S) = 1$

ii) $P(A) \geq 0$ FOR ALL $A \subset S$

iii) IF $A_1, A_2 \ldots$ ARE EVENTS $\ni A_i \cap A_j = \phi$ FOR ALL $i \neq j$. THEN $P(A_1 \cup A_2 \cup A_3 \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

IF A IS AN EVENT IN S

P (A) IS THE PROBABILITY OF A

PROB

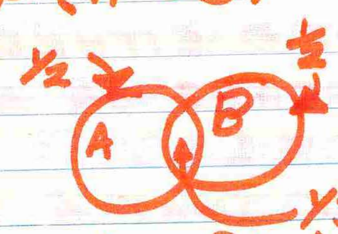
~~9/16/70~~ 9-17-70
9-18-70

EX) Pg 28 #7

$P(A)=\frac{1}{2}$; $P(B)=\frac{1}{2}$

$P(A\cup B)=2/3$

e) $P(\bar{A}\cup\bar{B})$



FIND $P(A\cap B)$

$P(A\cup B)= P(A)+P(B) - P(A\cup B)$

$+1/3 = + P(A\cap B)$.

$P(A\cap\bar{B}) = P(\bar{B}\cap A) = 1/6$

$P(\bar{A}\cup\bar{B}) = 1 - 1/3 = 2/3$ ETC.

$A_1, A_2 \ldots A_K \Rightarrow$ DISTINCT SINGLE ELE. EVENTS

$A = A_1 \cup A_2 \cup \ldots \cup A_K$

T.B.A. $P(A) = \sum_{n=1}^{k} P(A_j)$

PROOF

$A = A_1 + A_2 + \ldots + A_K$

WHERE $A_i \cap A_j = \phi \ni i \neq j$

$\therefore P(A) = P(A_1)+P(A_2)+\ldots+P(A_K)$

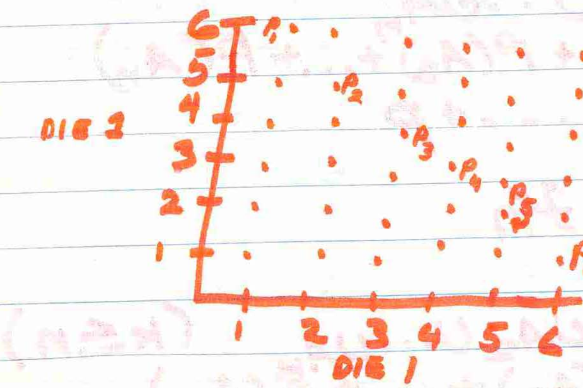
$\Rightarrow P(A) = \sum_{n=1}^{k} P(A_j)$

ACCOUNTABLY INFINITE SET: 1 TO 1 CORRES. TO POS. INTEGERS)

$A_1, A_2, A_3 \ldots$

$k\to\infty \ni$ AXIOM III WILL HOLD

(AN ACCOUNTABLE INFINITY)



$P(A) = P_1+P_2+P_3+P_4+P_5+P_6 = \frac{6}{36} = \frac{1}{6}$

COROLLARY:

1) LET A BE AN EVENT IN A SAMPLE SPACE S WITH n POINTS.

2) SUPPOSE P IS THE PROBABILITY MEASURE ON S $\ni$ EVERY SINGLE ELEMENT EVENT IN S IS EQUALLY LIKELY (i.e. HAS THE SAME PROBABILITY)

3) LET k OF THE POINTS IN S BE A. THEN $P(A)=k/n$

(OVER FOR PROOF)

(CONT.) PROOF

SUPPOSE $A_1, A_2 \ldots A_n$ ARE THE SINGLE ELEMENT EVENTS IN S.

$$S = A_1 \cup A_2 \cup \cdots \cup A_n$$
$$P(S) = P(A_1) + P(A_2) + \ldots + P(A_n)$$
$$1 = p + p + p + \ldots + p$$
$$1 = np \Rightarrow p = 1/n$$

SUPPOSE: $A = A_1 \cup A_2 \cup \ldots \cup A_k$ $(k \leq n)$
$$p(A) = P(A_1) + P(A_2) + \ldots + P(A_k)$$
$$= \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}$$
$$= k/n$$

9-20-70

DEFINITION: IF A FIRST OPERATION CAN BE PERFORMED IN ANY ONE OF n_1 WAYS AND A SECOND OPERATION CAN BE PERFORMED IN ANY OF n_2 WAYS, THEN BOTH OPERATIONS (THE SECOND IMMEDIATELY FOLLOWING THE FIRST) CAN BE PERFORMED IN $n_1 \times n_2$ WAYS

DEFINITION: AN ARRANGEMENT OF n SYMBOLS IN A DEFINITE ORDER IS CALLED A PERMUTATION OF THE n SYMBOLS (AN ARANGEMENT)

EX) 1, 2, 3

WRITE DOWN ALL PERMUTATIONS

1, 2, 3   3, 1, 2
2, 1, 3   1, 3, 2
2, 3, 1   3, 2, 1

DEFINITION: THE NUMBER OF DISTINCT (DIFFERENT) PERMUTATIONS (ARRANGEMENTS) OF r SYMBOLS SELECTED FROM A SET OF n DIFFERENT SYMBOLS $(n \geq r)$ EACH USED ONLY ONCE IS CALLED THE # OF PERMUTATIONS OF n THINGS TAKEN r AT A TIME. DENONTED BY P_r^n

EX.

$n = 1, 2, 3$

1, 2
2, 1
1, 3
3, 1
2, 3
3, 2 $\Rightarrow P_2^3 = 6$

THEOREM: $P^n_r = \frac{n!}{(n-r)!}$

PROOF: $\underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ SYMBOLS}}$

$$P^n_r = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!}$$

$$= n!/(n-r)!$$

USING n DIFFERENT SYMBOLS

COROLLARY: $P^n_n = n!$

PROOF: $P^n_n = \frac{n!}{0!} =$

EX) HOW MANY LISCENCE PLATES USING 4 #'S FROM $0,1,2,3,4,5,6,7,8,9$

$10\cdot10\cdot10\cdot10 = 10^4$ COMBINATIONS

DEFINITION: THE NUMBER OF DIFFERENT SUBSETS, EACH OF SIZE r, THAT A SET OF n DIFFERENT ELEMENTS HAS IS CALLED THE NUMBER OF n THINGS TAKEN r AT A TIME AND IS DENOTED BY C^n_r or $\binom{n}{r}$. ALSO $C^n_r = 1$ IF $r=0$. AND $C^n_r = 0$ IF $r>n$ OR $r<0$

9-21-70

1,2,3

$C^3_2 = 3;\ (1,2)(1,3)(2,3)$

THEM. $C^n_r = \frac{n!}{r!(n-r)!}$ FOR $0 \le r \le n$

$$P^3_2 = C^3_2 \cdot P^2_2$$

OR $P^n_r = C^n_r \cdot P^r_r$

$$\frac{n!}{(n-r)!} = C^n_r\, r! \Rightarrow C^n_r = \frac{n!}{r!(n-r)!}$$

EX) $C^3_2 = \frac{3!}{2!(3-2)!} = 3$

$$C^{1000}_{100} = \frac{1000!}{900!\,100!}$$

$$C^{52}_{13} = \frac{52!}{13!\,39!} \cong 6.35 \times 10^{11}$$

BINOMIAL THEOREM

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \dots + y^n$$

$$\frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{r!\,(n-r)!} = \frac{n!}{r!(n-r)!}$$

(CONT.)

$$\cancel{X}\ (x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{r}x^{n-r}y^r + \dots + \binom{n}{n}y^n$$

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

THEM: $\binom{n}{r} = \binom{n}{n-r}$

PROOF: $\binom{n}{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$

THEM: A SET WITH n ELEMENTS HAS 2^n SUBSETS

PROOF:

$$\#\text{OF SUB} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} + \dots + \binom{n}{n}$$
$$= \sum_{r=0}^{n} \binom{n}{r} = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} 1^r$$
$$= (1+1)^n = 2^n$$ (FROM BINOM. THEM.)

THEM: THE # OF DISTINCT PERMUTATION OF n OBJECTS
r_1 OF WHICH ARE IDENTICAL,
r_2 OF WHICH ARE " " ,
... r_K " " " " $(r_1 + r_2 + r_3 + \dots r_K = n)$
IS $n!/(r_1!\,r_2!\,r_3! \dots r_K!)$

9-25-70

PG. 43; #10

$P(A) = n(A)/n(S)$

$n(S) = \#$ OF POKER HANDS $= \binom{52}{5}$
$= \binom{52}{5}$

a) $\dfrac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$ (2 PAIRS)

b) FULL HOUSE

$\dfrac{\binom{13}{2}\binom{4}{3}\binom{4}{2}\binom{2}{1}}{\binom{52}{5}}$

$\binom{52}{5} \to$ # OF POSSIBLE POKER HANDS

OR $\dfrac{13 \cdot 12 \binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \dfrac{P^{13}_{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$

d) A STRAIGHT

A, 2, 3, 4, 5
2, 3, 4, 5, 6
⋮
10, J, Q, K, A

→ 10 DIFFERENT STRAITS

$\dfrac{10 \cdot \cancel{4}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}}$

TO GIVE A STRAIGHT FLUSH:

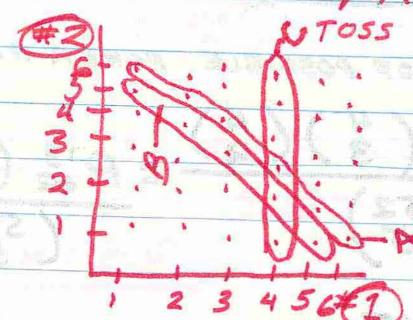
$$\frac{(10)(4)}{\binom{52}{5}}$$ ← # OF SUITS

(9-25-70)

~~FLUSH~~ STRAIGHT PROBABILITY

$10 \cdot 4^5$ POSSIBILITIES

CONDITIONAL PROBABILITY



LET A = 7 TOTAL $= \frac{6}{36}$

B = 4 ON DIE #1 $= \frac{6}{36}$

C = TOTAL OF 6 $= \frac{5}{36}$

$P(A \text{ GIVEN } B) = P(A/B) = 1/6$

$P(C/B) = 1/6$

$P(B/C) = 1/5 = \frac{n(B\cap C)}{n(C)}$

$$\Rightarrow P(X/Y) = \frac{n(X\cap Y)}{n(Y)}$$

DERIVATION:

$$P(X/Y) = \frac{\left[\frac{n(B\cap C)}{n(S)}\right]}{\left[\frac{n(C)}{n(S)}\right]} = \frac{P(B\cap C)}{P(C)}$$

DEF: LET A & B BE EVENTS IN A SAMPLE SPACE S AND LET $P(A) > 0$. THEN THE CONDITIONAL PROBABILITY OF B OCCURING GIVEN THAT A HAS OCCURED IS GIVEN BY:

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

THEOREM $P(A\cap B) = P(A)\,P(B/A)$

A) P(2 ACES IN 2 DRAWS WITHOUT REPL.) $= \frac{\binom{4}{2}\binom{48}{0}}{\binom{52}{2}} = \frac{12}{52\cdot 51}$

B) USEING ABOVE THEOREM:

P(2 ACE) = P(ACE ON 1'ST AND ON 2^{nd} DRAW)

($P(A\cap B)$)

= P(ACE ON FIRST DRAW) P(ACE ON SECOND DRAW GIVEN A)

$= P(A)\cdot P(B/A) = \frac{4}{52}\cdot\frac{3}{51} = \frac{12}{52\cdot 51}$

9-28-70

CONDITIONAL PROB. (CONT.)

THEM:

$P(A \cap B \cap C) = P(A)P(B/A)P(C/B \cap C)$

PROOF:

$P(A \cap B \cap C) = P(A \cap B)P(C/A \cap C)$

$= P(A)P(B/A)P(C/A \cap C)$

EX) PROBABILITY OF 3 ACES IN 3 DRAWS WITHOUT REPLACEMENT

= P(ACE ON FIRST DRAW <u>AND</u> ACE ON SECOND DRAW <u>AND</u> ACE ON THIRD DRAW)

$= \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50}$

EX) P(3 ACES IN 3 DRAWS)

$= \frac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}} = \frac{4}{\frac{52!}{3!49!}} = \frac{4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50}$ SAME AS ABOVE

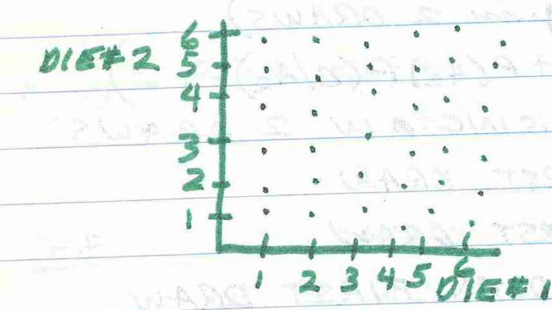
EX) P(ACE & KING IN TWO DRAWS)

= P(ACE ON FIRST DRAW <u>AND</u> KING ON FIRST DRAW <u>OR</u> KING ON FIRST DRAW <u>AND</u> ACE ON SECOND DRAW)

$P(ACE, 1\ KING) = \frac{4}{52} \cdot \frac{4}{51} + \frac{4}{52} \cdot \frac{4}{51}$

$\Rightarrow P(A \cup B) = P(A) + P(B)$ WHEN A & B ARE DISJOINT

<u>DEFINITION</u>: LET $A_1, A_2 \ldots A_k$ BE EVENTS IN A SAMPLE SPACE $S \ni A_i \cap A_j = \phi$ FOR ALL $A_i \neq A_j$ AND $S = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k$. THEN $A_1, A_2, A_3 \ldots A_k$ FORM A PARTITION OF THE SAMPLE SPACE S.



$\Rightarrow$ SO WHAT "?#*$!!"

<u>THEM</u>. LET $A_1, A_2, \ldots A_k$ FORM A PETITION OF THE SAMPLE SPACE S. LET B BE AN EVENT IN S. THEN

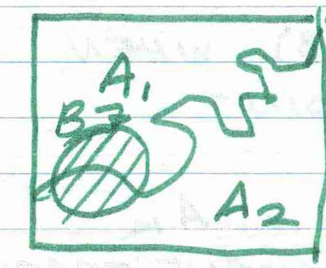
$P(B) = P(B) + P(B/A_1) + P(A_2)P(B/A_2) + \ldots + P(A_k)P(B/A_k)$

<u>PROOF FOR K=2</u>: FOR K=2 THE THEOROM BECOMES:

$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2)$

(CONT.)

SAMPLE SPACE



$B = (A_1 \cap B) \cup (A_2 \cap B)$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$

$= P(A_1)P(B/A_1) + P(A_2)P(B/A_2)$

9-26-70

EX) P(1 ace & 1 king in 2 DRAWS)

($P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) \ldots$)

LET B = 1 ACE & 1 KING IN 2 DRAWS

A_1 = ACE ON FIRST DRAW

A_2 = KING ON FIRST DRAW

A_3 = ANOTHER CARD ON FIRST DRAW

A_1, A_2, A_3 FORM A PARTITION OF SAMPLE SET

FITTING INTO THEOREM:

$P(B) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + P(A_3)P(B/A_3)$

$= \frac{4}{52}\cdot\frac{4}{51} + \frac{4}{52}\cdot\frac{4}{51} + \frac{44}{52}\cdot 0$

$[P(A_1) + P(A_2) + P(A_3) = 1]$

$B = \frac{32}{52\cdot 51}$

(EX)

AUXILERY PROBLEMS

1) D = A PART IS DEFECTIVE

A = A PRODUCED PART

B = B " "

C = C " "

GIVEN $P(A) = 3/10 = .3$, $P(B) = 4/10 = .4$, $P(C) = 3/10 = .3$ → PARTITION

$P(D/A) = .01$

$P(D/B) = .02$

$P(D/C) = .05$

$P(D) = P(D/A)P(A) + P(D/B)P(B) + P(D/C)P(C)$

$= (.01)(.3) + (.02)(.4) + (.05)(.3)$

THEOREM: (BAYES' THEOREM)

LET $A_1, A_2, A_3 \ldots A_k$ FORM A PARTITION OF THE SAMPLE SPACE S. LET B BE AN EVENT IN S, AND LET j BE ONE OF THE INTEGERS $1, 2, 3 \ldots k$. THEN

$$P(A_j/B) = P(B/A_j)\cdot P(A_j) \Big/ \sum_{i=1}^{k} P(B/A_i)P(A_i)$$

PROOF:

$$P(A_j/B) = \frac{P(B \cap A_j)}{PB} = \frac{P(A_j)\cdot P(B/A_j)}{P(A_1)P(B/A_1)\cdots}$$

$$= \frac{P(B/A_j)P(A_j)}{\sum_{i=1}^{k} P(B/A_j)P(B/A_j)}$$

(EX) AUX. PROBLEMS ?

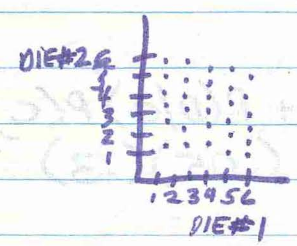
11b) $P(A/D) = \dfrac{P(A)\,P(D/A)}{\underbrace{P(A)(P(D/A)+P(B)P(D/B) + P(C)P(D/C)}_{P(\text{OF A DEFECTIVE PART})}}$

$P(A/D) = \dfrac{(.3)(.01)}{(.3)(.01)+(.4)(.02)+(.3)(.05)}$

$= \dfrac{.003}{.026} = 3/26$

SIMILARLY: $P(B/D) = 8/26$

$P(C/D) = 15/26$



A = TOTAL 7

B = 4 ON DIE 1

C = TOTAL OF 6

D = 4 ON DIE 2

DEFN: LET A AND B BE EVENTS IN A SAMPLE SPACE S. THEN A & B ARE IND. IFF $P(A \cap B) = P(A)\,P(B)$

THEOREM: LET $P(A) \neq 0$. THEN A AND B ARE INDEPENDENT IFF $P(B/A) = P(B)$

2 PART PROOF: ASSUME A & B ARE IND.

$P(B/A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{P(A)\,P(B)}{P(A)} = P(B)$

ASSUME $P(B/A) = B$

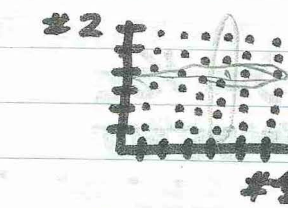
ETC. ETC.

10-1-70

IFF A & B ARE INDEP

$P(A \cap B) = P(A)\,P(B)$

EXAMPLE OF INDEP. EVENTS



A: TOTAL OF 7

B: 4 ON DIE #1

C: TOTAL OF 6

D: 4 ON DIE #2

B & D ARE SEEMINGLY INDEPENDENT

$P(B \cap D) = 1/36$

$P(B) = P(D) = \frac{1}{6}$

$P(B \cap D) = P(B)\,P(D)$

∴ THEY IS INDEPENDENT

$P(D) = 1/6$

$P(B/D) = 1/6$

∴ B & D IS AGAIN INDEP

TAKE B & C

$P(B) = 1/6$

$P(C) = 5/36$

$P(B \cap C) = 1/36$

$P(B \cap C) \neq P(B)\,P(C)$

NOT INDEPENDENT (CONT.)

$P(B) = 1/6$

~~[illegible]~~ $P(B/C) = 1/5$

B & C ARE NOT INDEPENDENT

CONSIDER A & B

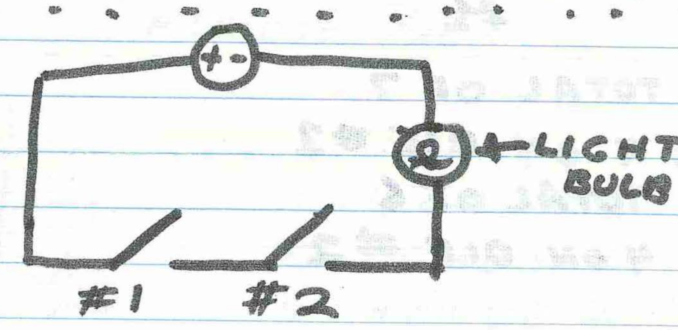
$P(A) = P(B) = 1/6$

$P(A)P(B) = P(A \cap B) = 1/36$

THESE ARE INDEPENDENT

. . . .

EX)



LET A BE EVENT #1 CLOSES

LET B BE EVENT #2 CLOSES

$P(A) = .9$

$P(B) = .8$

ASSUME A & B ARE INDEP. EVENTS

$P(\text{CIRCUIT WORKS}) = P(C)$

$P(C) = P(A \cap B) = P(A)P(B) = .72$

$P(\text{CIRCUIT FAILS}) = 1 - P(C)$

$= .28$

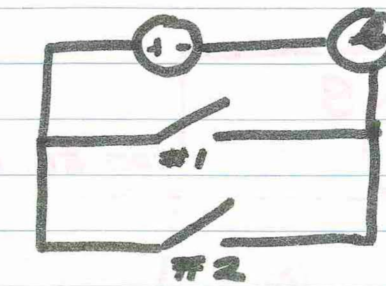
ALSO $P(\text{CIRCUIT FAILS}) = P(\bar{A} \cup \bar{B})$

$P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$

$= P(\bar{A}) + P(\bar{B}) - P(\bar{A})P(\bar{B})$

$= (.1) + (.2) - (.1)(.2) = .28$

EX)



SWITCHES INDEPENDENT

A = SWITCH 1 CLOSD

B = " 2 "

$P(A) = .9$

$P(B) = .8$

$P(\text{CIRCUIT WORKS}) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)P(A)$

$= .98$

. . . .

1) **DEFN.** LET A, B, & C BE EVENTS IN A SAMPLE SPACE S. THEN A, B, & C ARE PAIRWISE IND. IFF

$P(A \cap B) = P(A)P(C)$

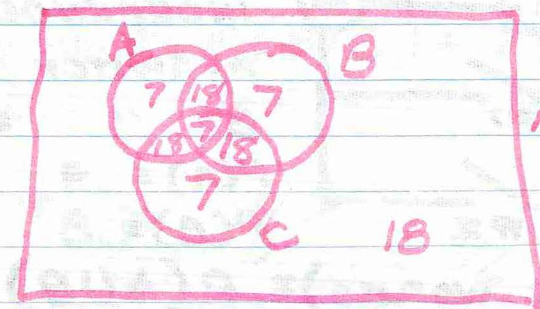
$P(B \cap C) = P(B)P(C)$

$P(A \cap C) = P(A)P(C)$

2) **DEFN:** LET A, B, & C BE EVENTS IN A SAMPLE SPACE S. THEN A, B, & C ARE IND. IFF A, B, & C ARE PAIRWISE IND. AND

$P(A \cap B \cap C) = P(A)P(B)P(C)$

10-2-70



100 EL. EVENTS.

$P(A)=\frac{1}{2}$; $P(B)=\frac{1}{2}$; $P(C)=\frac{1}{2}$

$P(A\cap B)=1/4$; $P(A\cap C)=\frac{1}{4}$; $P(B\cap C)=1/4$
$=P(A)P(B)$; $=P(A)P(C)$; $=P(B)P(C)$

$\therefore$ A, B, & C ARE PAIRWISE INDEPENDENT

$P(A\cap B\cap C)= 7/100$

IS $P(A\cap B\cap C)\stackrel{?}{=} P(A)P(B)P(C)$
$7/100 \neq 1/8$

$\therefore$ A, B, & C ARE NOT INDEPENDENT

DEFN: TWO EVENTS ARE MUTUALLY EXCLUSIVE IFF $A\cap B=\emptyset$.
ie) A & B ARE DISJOINT

THEM: LET $P(A)>0$ AND $P(B)>0$, IF A & B ARE MUTUALLY EXCLUSIVE THEY ARE NOT INDEPENDENT AND IF A & B ARE INDEP. THEY ARE NOT MUTUALLY EXCLUSIVE.

(CONT.)

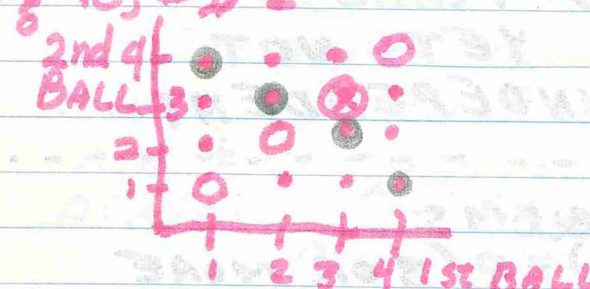
PROOF: ASSUME A & B ARE MUTUALLY EXCLUSIVE & INDEPENDENT

$P(A\cap B)=0$
$P(A)P(B)=0$
EITHER $P(A)=0$ OR $P(B)=0$.
HOWEVER $(A \text{ \& } B>0)$

Pg 42; #4 A = # OF ARANGEMENTS
$n=20$
$r_1=15$ (RED)
$r_2=5$ (GREEN)

$$A=\frac{n!}{r_1!\,r_2!}=\frac{20!}{15!\,5!}=\binom{20}{5}$$

Pg 48; #1



P(ANY EVENT) $=1/12$ $P(B_1)=1/4$
A: SUM OF 5; $P(A)=4/12=1/3$
B_i: i ON FIRST BALL

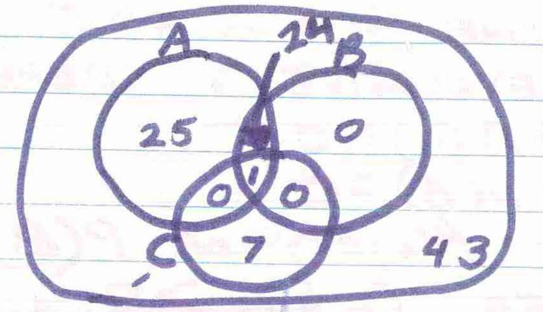
$$P(A/B_1)=\frac{n(A\cap B_1)}{n(B_1)}=1/3$$

$P(A/B_i)=1/3$ $i=1,2,3,4$

$P(B_1/A)=1/4=P(B_1)=1/4$
$\therefore$ A & B_i ARE INDEPENDENT

26

10-5-70



$P(A\cap B\cap C) = 1/100$

$P(A) = 50/100 = 1/2$

$P(B) = 1/4$

$P(C) = 8/100$

$P(A)P(B)P(C) = \frac{1}{2}\cdot\cancel{\frac{1}{4}\cdot\frac{8}{100} = \frac{1}{100}} = P(A\cap B\cap C)$

$P(A\cap B) = 1/4$

$P(A)P(B) = 1/8$

$P(A)P(B) \neq P(A\cap B)$

$\Rightarrow$ INDEPENDENT YET NOT PAIRWISE INDEPENDENT

- - - - - - - - - - - - - - - - - - - -

AUXILERY PROBLEMS

10) a) P(CRASH) = P(3 OR MORE ENGINES FAILING)
= P(EXACTLY THREE ENGINES OR EXACTLY 4 ENGINES FAILING)

THIS EVENT IS MUTUALLY EXCLUSIVE

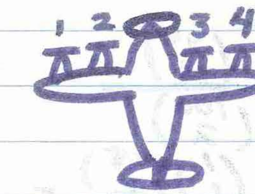
$\therefore$ P(CRASH) = P(EXACTLY 3 FAILING) + P(EXACTLY 4 ENGINES FAILING)

P(4 ENGINES FAILING) = $(.05)^4$

P(3 ENGINES FAILING) = $.95(.05)^3 4$

(CONT)

b) P(CRASH)



1 & 2 FAIL

3 & 4 FAIL

$\Rightarrow$ P(CRASH) = P(1 & 2 FAILING OR 3 & 4 FAILING)

= P(1 & 2 FIAL) + P(3 & 4 FAIL) − P(1 & 2 FAIL AND 3 & 4 FAIL)

$= (.05)^2 + (.05)^2 - (.05)^4$

42) SHOULD EMPLOY BAYE'S THEM.

A = FROM MACHINE A

B = " " B

C = " " C

D = DEFECTIVE

WANT TO FIND P(A/D)

A, B, & C FORM PARTICIAN OF S.S.

$$P(A/D) = \frac{P(D/A)P(A)}{P(D/A)P(A) + P(D/C)P(B) + P(D/C)P(C)}$$

P(A) = .25

P(B) = .35

P(C) = .40

P(D/A) = .05

P(D/B) = .04

P(D | C) = .02

PLUG &

CHUG

Pg 42

6) a) $\dfrac{\binom{99}{1}}{\binom{100}{2}}$

b) $\dfrac{\binom{1}{1}\binom{99}{0}}{\binom{100}{2}} = \dfrac{1}{\binom{100}{2}}$

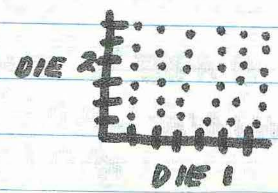
10-8-70

IN CHAPT. 3 SEC. 3.1

RANDOM VARIABLES

FROM EXPERIMENTS

EX)



LET X BE THE TOTAL ON THE TWO DICE

ω	VALUE OF X AT ω
(2,2)	4
(5,6)	11
⋮	

WILL GET A REAL VALUED FUNCTION

DEF: A RANDOM VARIABLE X IS A REAL VALUED FUNCTION OF THE ELEMENTS OF A SAMPLE SPACE S. AN OUTCOME x ~~(X)~~ IS A VALUE OF X.

$$x = X(\omega)$$

EX) FROM DIE PBLM.

$$X((5,6)) = 11$$
$$X((2,3)) = 5$$

REMARK: IF $\omega \in S$, WE CAN USE THE NOTATION $x = X(\omega)$, TO REPRESENT THAT x IS OUTCOME OF X AT ω. HOWEVER, WE ORDINARILY USE THE NOTATION $P(X=x)$ TO MEAN THE PROBABILITY X HAS OUTCOME.

E) FROM DIE PROB.

$$P(X=7) = 1/6$$
$$P(X=6) = 5/36$$
$$P(X=2) = 1/36$$

DEF: A RANDOM VARIABLE X IS DISCRETE IF ITS SET OF OUTCOMES IS FINITE OR IS ACCOUNTABLY INFINITE

EX) LET Y BE THE NUMBER OF THE FLIPS NECESSARY TO GET A HEAD (ACCOUNABLY INFINITE)

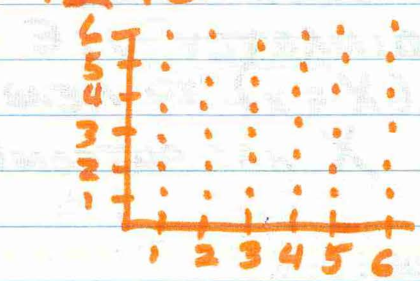
X IS CONTINUOUS IF ITS SET OF OUTCOMES IN AN UNCOUNTABLY INFINITE SET OF REAL NUMBERS AND $P(X=x)=0$, FOR x ANY ONE OF POSSIBLE OUTCOMES OF X.

DEFN: LET X BE A DISCRETE RANDOM VARIABLE, THEN IT'S PROBABILITY FUNCTION IS GIVEN BY:

$$g(x) = P(X=x)$$

10-12-70

P(CONTINUOUS RANDOM VARIABLE) $=0$



LET X BE THE SUM OF SPOTS ON THE TWO DICE

x	$p(x) = P(X=x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

(CONT.)

$$\sum_{\text{ALL POSIBLE X}} P(x) = 1$$

USEFUL TO REPRESENT FUNCTION AS AN EQUATION:

$$P(x) = \begin{cases} \frac{x-1}{36} & \text{FOR 2 TO 7 } (2 \le x \le 7) \\ \frac{13-x}{36} & \text{FOR 7 TO 12 } (7 \le x \le 12) \end{cases}$$

LET Y BE THE NUMBER OF HEADS WHEN COIN IS TOSSED THRICE

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Y	vs $P(y) = P(Y=y)$
0	1/8
1	3/8
2	3/8
3	1/8

$$p(y) = \frac{\binom{3}{y}}{8} \text{ FOR } y=0,1,2,3$$

DEFN: LET X BE A CONTINUOUS RANDOM VARIABLE. IF $\exists$ A FUNCTION $f \ni$

a) $f(x) \ge 0 \ \forall x$

b) $\int_{-\infty}^{\infty} f(x)dx = 1$

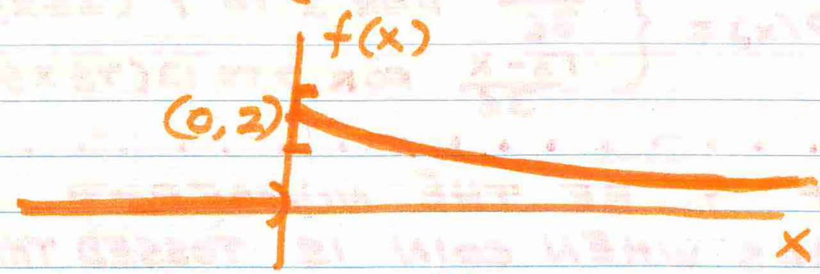
c) $P(a < X < b) = \int_a^b f(x)dx \ \forall a<b$

THEN f IS CALLED THE DENSITY FUNCTION FOR X

PROB

EXAMPLE LET $\underline{X}$ HAVE DENSITY FUNCTION ~~f~~

$$f(x) \begin{cases} = 2e^{-2x} & \forall\ x \geq 0 \\ = 0 & \forall\ x < 0 \end{cases}$$



① $f(x) \geq 0\ \forall x$

② $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} 2e^{-2x}\,dx$

$= 1$

$P(1 < \underline{X} < 2) = \int_1^2 2e^{-2x}\,dx$

$= e^{-2} - e^{-4}$

$P(-1 < \underline{X} < 2) = P(0 < x < 2)$

$= \int_0^2 2e^{-2x}\,dx$

$= 1 - e^{-4}$

$P(-1 < x \leq 2) = P(-1 < x < 2)$

$\because P(2) = 0$

10-13-70

Pg 68

5) 3-19
4-20
1-21
1-24
1-26

	P	$\overline{X}$
19 & 19 →	$\frac{3}{10} \cdot \frac{2}{9} = 6/90$	19
19 & 20 →	$2 \cdot \frac{3}{10} \cdot \frac{4}{9} = 24/90$	19.5
19 & 21 →	$2 \cdot \frac{3}{10} \cdot \frac{1}{9} = 6/90$	20
19 & 24 →	$2 \cdot \frac{3}{10} \cdot \frac{1}{9} = 6/90$	21.5
19 & 26 →	$2 \cdot \frac{3}{10} \cdot \frac{1}{9} = 6/90$	22.5
20 & 20 →	$\frac{4}{10} \cdot \frac{3}{9} = 12/90$	20
20 & 21 →	$2 \cdot \frac{4}{10} \cdot \frac{1}{9} = 8/90$	20.5
20 & 24 →	$2 \cdot \frac{4}{10} \cdot \frac{1}{9} = 8/90$	22
20 & 26 →	$2 \cdot \frac{4}{10} \cdot \frac{1}{9} = 8/90$	23
21 & 24 →	$2 \cdot \frac{1}{10} \cdot \frac{1}{9} = 2/90$	22.5
21 & 26 →	$2 \cdot \frac{1}{10} \cdot \frac{1}{9} = 2/90$	23.5
24 & 26 →	$\frac{1}{10} \cdot \frac{1}{9} \cdot 2 = 2/90$	25

$\Sigma P = 90/90 = 1$ (HURRAH!)

$\overline{X}$	19	19.5	20	20.5	21.5	22	22.5
$P(\overline{X})$	6/90	24/90	18/90	8/90	6/90	8/90	8/90

23	23.5	25
8/90	2/90	2/90

$\Sigma P(\overline{X}) = 1$

DEFN: LET X BE A RANDOM VARIABLE, THEN THE CUMULATIVE DISTRIBUTION FUNCTION FOR X IS GIVEN BY:

$$F_X(t) = P(X \le t) \text{ FOR } t = \text{ANY REAL \#}$$

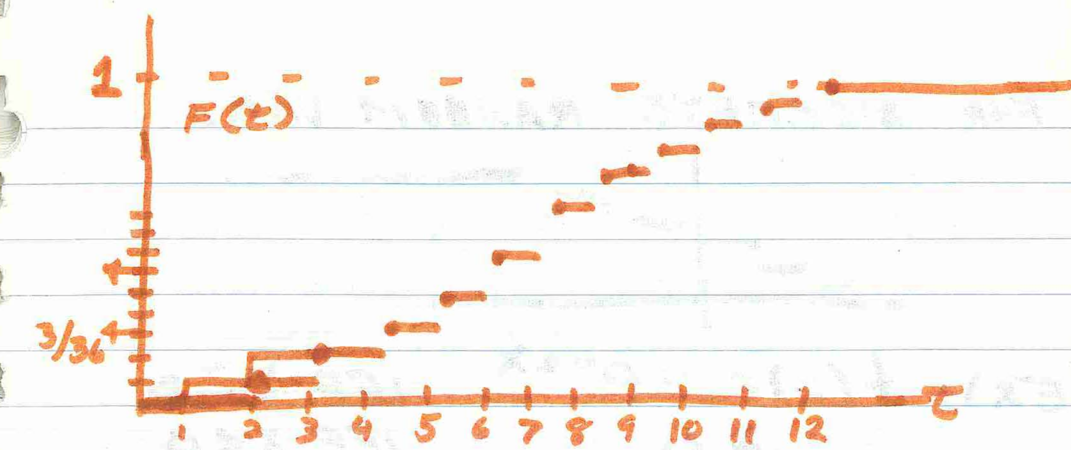
1) IF X IS DISCRETE WITH PROB. FUNCTION $p(x)$,
$$F_X(t) = \sum_{x \le t} p(x)$$

2) IF X IS CONTINUOUS WITH $f(x)$, $F_X(t) = \int_{-\infty}^{t} f(x)\,dx$

EXAMPLE:
LET X BE # SPOTS ON 2 DICE

$$P(x) = \begin{cases} \frac{x-1}{36}; & x = 1,2,3,4,5,6,7 \\ \frac{13-x}{36}; & x = 7,8,9,10,11,12 \end{cases}$$

$$F_X(t) = P(X \le t) = \begin{cases} 0 & \text{FOR } t < 2 \\ 1/36 & \text{FOR } 2 \le t < 3 \\ 3/36 & \text{FOR } 3 \le t < 4 \\ 6/36 & \text{FOR } 4 \le t < 5 \\ 10/36 & \text{FOR } 5 \le t < 6 \\ 15/36 & \text{FOR } 6 \le t < 7 \\ 21/36 & \text{FOR } 7 \le t < 8 \\ 26/36 & \text{FOR } 8 \le t < 9 \\ 30/36 & \text{FOR } 9 \le t < 10 \\ 33/36 & \text{FOR } 10 \le t < 11 \\ 35/36 & \text{FOR } 11 \le t < 12 \\ 1 & \text{FOR } 12 \le t < \infty \end{cases}$$



1) CUM. DISTRIBUTION FOR A DISCRETE RANDOM VARIABLES IS A STEP FUNCTION

2) $\lim_{t \to +\infty} F_x(t) = 1$

3) $\lim_{t \to -\infty} F_x(t) = 0$

4) IF $t_1 > t_2$, $F_x(t_1) \ge F_x(t_2)$

2, 3, 4 $\Rightarrow$ CUM. DIST. FUNCTION

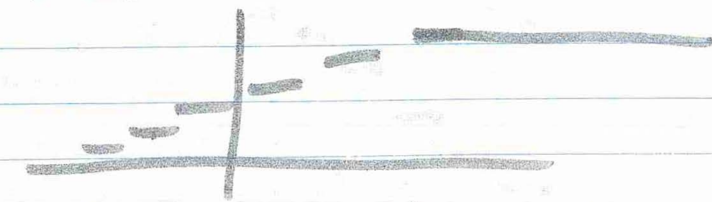
10-19-70

$\lim_{t \to \infty} F_x(t) = 1$

$\lim_{t \to -\infty} F_x(t) = 0$

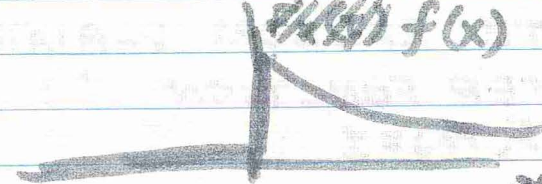
$F_x(t) = P(x \le t)$
A STEP FUNCTION FOR DISCRETE VALUES.

FOR DISCRETE RANDOM VAR.



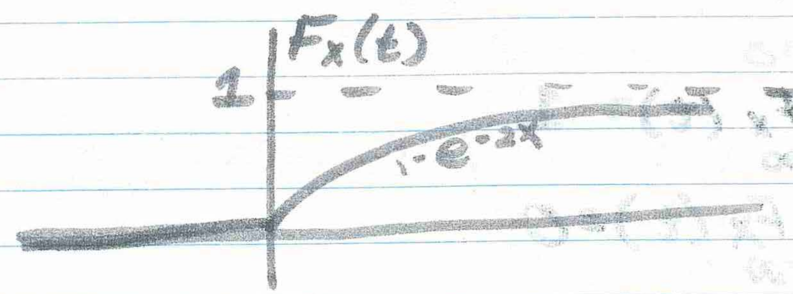
EX) $f(x) = 2e^{-2x}$ IFF $x \geq 0$
$= 0$ IFF $x < 0$

THIS IS A CONT. RANDOM VAR.



$$F_x(t) = P(x \leq t) = \int_{-\infty}^{x} f(x)\,dx$$

$$F_x(t) = 0 \quad \text{FOR } t<0$$
$$= \int_0^t 2e^{-2x}\,dx \quad \text{FOR } t<0$$
$$= 1 - \cancel{2}e^{-2t} \quad " \;\; " \;\; "$$



SAME PROPERTIES AS DISCRETE, FURTHERMORE, IT'S CONTINUOUS

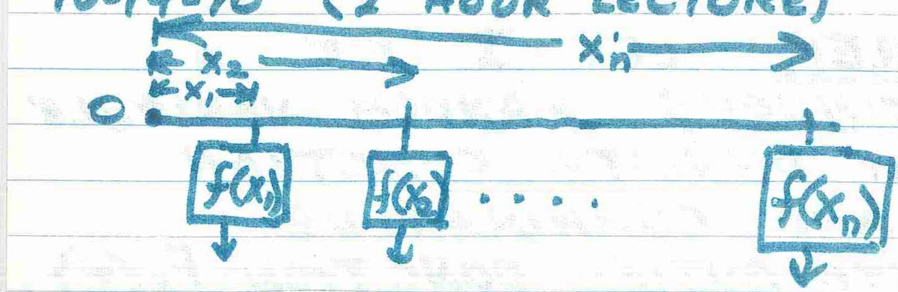
THEOREM: LET X BE A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION f AND CUMMULATIVE DISTRIBUTION FUNCTION $F_X(t)$. THEN $F_X(t)$ IS DIFFERENTIABLE EVERYWHERE EXCEPT PERHAPS AT A FINITE NUMBER OF POINTS IN A FINITE INTERVAL, AND

$$f(t) = \frac{d}{dx}\left[F_X(t)\right]$$

EX) (FROM PREV. PAGE)

$$f(t) = \frac{d}{dt}\left[F_x(t)\right] = \begin{cases} \frac{d}{dt}(0) = 0; & t<0 \\ \frac{d}{dt}(1-e^{-2t}) = 2e^{-2t} & t>0 \end{cases}$$

10-19-70 (2 HOUR LECTURE)



LOOK AT FIRST MOMENT WITH RESPECT TO THE ORIGIN

$$= x_1 f(x_1) + x_2 f(x_2) + \ldots + x_n f(x_n)$$

$$= \sum_{i=1}^{n} x_i f(x_i)$$

FIND μ (CENTER OF GRAVITY)

$$\mu = \frac{\sum_{i=1}^{n} x_i f(x_i)}{\sum_{i=1}^{n} f(x_i)} \quad (= \text{CENTROID})$$

LET $\sum_{i=1}^{n} f(x_i) = 1$

THEN $\mu = \sum_{i=1}^{n} x_i f(x_i)$

IF X IS DISCRETE WITH PROBABILITY FUNCTION $p(x)$;

$$E(X) = \sum_{\text{ALL } x} x\, p(x)$$

NOTICE ALALOGY BETWEEN CENTER OF GRAVITY AND RANDOM VARIABLE MEAN

DEFN. LET H BE A FUNCTION AND LET X BE A RANDOM VARIABLE. $H(X)$ IS A RANDOM VARIABLE WHOSE OUTCOME IS $H(x)$ WHENEVER X HAS AN OUTCOME x

EX) 1) $H(X) = X^2$

2) $H(X) = e^H$

DEFN. LET ~~X~~ $H(X)$ BE A FUNCTION OF THE RANDOM VARIABLE X

1) IF X IS DISCRETE WITH PROBABILITY FUNCTION $p(x)$

$$E[H(X)] = \sum_{\text{ALL } x} H(x) p(x)$$

PROVIDED THIS SUM IS ABSOLUTELY CONVERGENT

2) IF X IS CONTINUOUS WITH DENSITY FUNCTION $f(x)$,

$$E[H(x)] = \int_{-\infty}^{\infty} H(x) f(x)\, dx$$

PROVIDED THE INTEGRAL IS ABSOLUTELY CONVERGENT (IF EITHER NOT CONVERGENT, DON'T EXIST.)

<u>DEFN</u>: $\mu_x = E(X)$ IS CALLED THE MEAN OF THE RANDOM VARIABLE X

<u>THEM</u>: IF X IS A RANDOM VARIABLE AND C IS A CONSTANT

1) $E(C) = C$
2) $E[C\,H(X)] = C \cdot E[H(X)]$
3) $E[H(x) + G(x)] = E[H(x)] + E[G(x)]$

PROOF: (HOT DING!)

PROOF FOR X CONTINUOUS WITH DENSITY FUNCTION $f(x)$

1) $E(C) = \int_{-\infty}^{\infty} C f(x)\,dx$
$= C \int_{-\infty}^{\infty} f(x)\,dx$
$= C$

2) $E[C\,H(X)] = \int_{-\infty}^{\infty} C\,H(x) f(x)\,dx$
$= C \int_{-\infty}^{\infty} H(x) f(x)\,dx$
$= C\,E[H(X)]$

3) $E[H(X) + G(X)]$
$= \int_{-\infty}^{\infty} [H(x) + G(x)] f(x)\,dx$
$= \int_{-\infty}^{\infty} H(x) f(x)\,dx$
$+ \int_{-\infty}^{\infty} G(x) f(x)\,dx$
$= E[H(X)] + E[G(X)]$

EX) LET X BE THE TOTAL ON 2 DICE

x	$p(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

$E(X) = \sum_{\text{ALL } x} x\,p(x)$
$= 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{4}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36}$
$= \frac{252}{36}$
$= 7$ (HOW 'BOUT THAT!)

EX) $E[X^2]$ $(\neq E[X])$

$E[X^2] = \sum_{\text{ALL } x} x^2 p(x)$
$= 1974/36$

(CONT.)

(CONT)

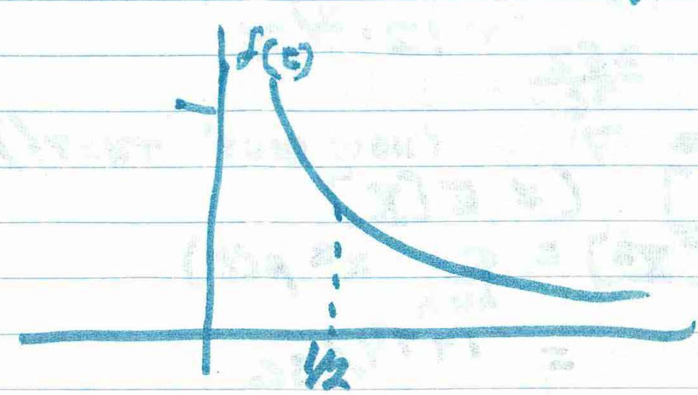
$E(X) = 7$

$E[X^2] = 1974/36$

LET X BE A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION:

$$f(x) = 2e^{-2x} \quad \text{FOR } x \geq 0$$
$$= 0 \quad x < 0$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$
$$= \int_{-\infty}^{0} 0 + \int_{0}^{+\infty} 2xe^{-2x} dx$$
$$= 2\int_{0}^{+\infty} x e^{-2x} dx$$
$$= e^{-2x}\left[\frac{x}{-2} - \frac{1}{4}\right]_0^{+\infty}$$
$$= 0 - 2e\left(\frac{0}{-2} - \frac{1}{4}\right) = \frac{1}{2}$$



WE WANT $m \ni \int_{-\infty}^{m} f(x)dx = 1/2$

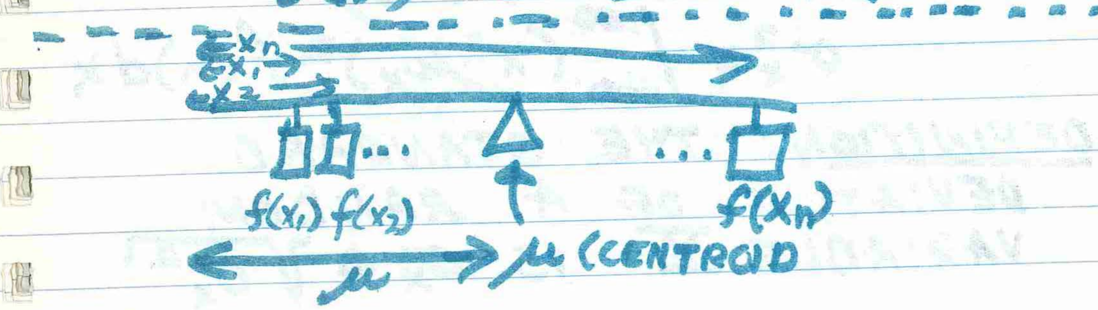
(DIVIDING AREA IN HALF)

$$\int_0^m 2e^{-2x} dx = 1/2$$
$$1 - e^{-2m} = 1/2$$
$$\Rightarrow m = \ln 2/2$$

m IS NOT MEAN $(E(X))$, BUT IS THE MEDIUM

FIND $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$$= \int_0^{\infty} x^2 \, 2e^{-2x} dx$$
$$= 2e^{-2x}\left[-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4}\right]_0^{\infty}$$
$$E(X^2) = 0 - \left[2\left(-\frac{1}{4}\right)\right] = \frac{1}{2}$$



$$(x_1 - \mu)^2 f(x_1) + (x_2 - \mu)^2 f(x_2) + \ldots$$
$$= \sum_{i=1}^{n} (x_i - \mu)^2 f(x_n)$$

THE MOMENT OF INERTIA

MOMENT OF INERTIA MEASURE OF HOW MUCH THE WEIGHTS ARE SPREAD APART

DEFINITION: LET X BE A RANDOM VARIABLE WITH MEAN μ_x, THEN THE VARIENCE OF X IS GIVEN BY

$$\sigma^2 = E\left[(X-\mu_x)^2\right]$$

REMARK: IF X IS DISCRETE WITH PROBABILITY FUNCTION $p(x)$,

$$\sigma_x^2 = \sum_{\text{ALL } x} (x-\mu_x)^2 p(x),$$

AND IF X IS CONTINUOUS WITH DENSITY FUNCTION $f(x)$,

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)\,dx$$

DEFINITION: THE STANDARD DEVIATION OF A RANDOM VARIABLE X IS $\sigma_x = \sqrt{\sigma_x^2}$

BACK TO DICE.

FROM BEFORE $\mu_x 7$

X	$X-\mu_x$	$(X-\mu_x)^2$	$p(x)$
2	-5	25	1/36
3	-4	16	2/36
4	-3	9	3/36
5	-2	4	4/36
6	-1	1	5/36
7	0	0	6/36
8	1	1	5/36
9	2	4	4/36
10	3	9	3/36
11	4	16	2/36
12	5	25	1/36

$$\sigma_x^2 = \sum_{\text{ALL } x} (x-\mu_x)^2 p(x)$$

$$= 25\tfrac{1}{36} + 16\tfrac{2}{36} + 9\tfrac{3}{36} + 4\tfrac{4}{36} + 1\tfrac{5}{36} + 0\tfrac{6}{36} + 1\tfrac{5}{36} + 4\cdot\tfrac{4}{36} + 9\ 3/36 + 16\ 2/36 \cdot 25\ 1/36$$

$= 210/36$

$= 35/36$

WELL WHUP WHUP!

<u>THEOREM</u>: $\sigma_x^2 = E(X^2) - \mu_x^2$

PROOF: $\sigma_x^2 = E[(X-\mu_x)^2]$

$$= E[X^2 - 2\mu_x X + \mu_x^2]$$
$$= E(X^2) + E(-2\mu_x X) + E(\mu_x^2)$$
$$= E(X^2) - 2\mu_x E(X) + \mu_x^2$$
$$= E(X^2) - 2\mu_x^2 + \mu_x^2$$
$$= E(X^2) - \mu_x^2$$

FOR BOTH DISCRETE & CONTINUOUS RANDOM VARIABLES

EXAMPLE: FROM PREVIOUS DICE PROBLEMS

$$E(X^2) = 1974/36$$
$$\mu_x = +7$$
$$E(X^2) - \mu_x^2 = \frac{1974}{36} - 49$$
$$= 210/36 = 35/36$$

WHICH (LO & BEHOLD!) CHECKS

<u>DEFINITION</u>: LET X BE A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION f. THE MEDIAN m OF X IS A NUMBER IS A NUMBER ∋

$$\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx$$

10-20-70

$$\sigma_x^2 = E[(x-\mu_x)^2] = \sum_{\text{ALL } x} (x-\mu_x)^2 p(x)$$
FOR DISCRETE X
$$= \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)\,dx$$
FOR CONTINUOUS X

σ_x = STANDARD DEVIATION

μ_x = MEAN OF DISTRIBUTION = $E(X)$

IN DATA ANALYSIS: $\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$

<u>THEM.</u> $\sigma_x^2 = E(X^2) - \mu_x^2$

<u>THEOREM</u>: LET X BE A RANDOM VARIABLE WITH CUMULATIVE DISTRIBUTION FUNCTION $F_X(t)$. LET $Y = a + bX$, WHERE $b > 0$. THEN THE CUMULATIVE DISTRIBUTION FUNCTION OF Y IS $F_Y(t) = F_X\left(\frac{t-a}{b}\right)$

PROOF: $F_Y(t) = P(Y \le t)$

$$= P(a + bX \le t)$$
$$= P(bX \le t - a)$$
$$= P\left(X \le \frac{t-a}{b}\right)$$
$$= F_X\left(\frac{t-a}{b}\right)$$

COROLLARY: IF X IS CONTINUOUS WITH DENSITY FUNCTION $f_X(x)$, THEN THE DENSITY FUNCTION OF Y IS $f_Y(y) = \frac{1}{b} f_X\left(\frac{y-a}{b}\right)$

PROOF: $F_Y(t) = F_X\left(\frac{t-a}{b}\right)$

$$f_Y(t) = \frac{d}{dt}[F_Y(t)] = \frac{d}{dt}\left[F_X\left(\frac{t-a}{b}\right)\right]$$

$$= \frac{1}{b} F_X'\left(\frac{t-a}{b}\right) \quad \text{BY CHAIN RULE}$$

$$= \frac{1}{b} f_X\left(\frac{t-a}{b}\right)$$

$$f_Y(y) = \frac{1}{b} f_X\left((y-a)/b\right)$$

EX $Y = 5+3X$ WHEN $f_X(X) = 2e^{-2x}\mu(t)$

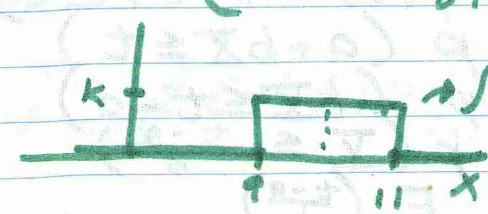
$$f_X\left(\frac{Y-5}{3}\right) = 2\,\mathrm{EXP}\left[-2\left(\frac{Y-5}{3}\right)\right] \quad \text{FOR } \frac{Y-5}{3} \ge 0$$
$$= 0 \quad \text{OTHERWISE}$$

$$f_Y(Y) = \frac{2}{3} e^{-2/3\,Y + 10/3} \quad \text{FOR } Y \ge 5$$
$$= 0 \quad \text{OTHERWISE}$$

AUX PROB # 58

LET X BE THE CURRENT

$$f(x) = \begin{cases} k & \text{FOR } 9 \le x \le 11 \\ 0 & \text{OTHERWISE} \end{cases}$$



$\int = 1 \Rightarrow k = 1/2$

(CONT.)

(CONT.)

$$\therefore f(x) = \begin{cases} \frac{1}{2} & \text{FOR } 9 \le x \le 11 \\ 0 & \text{OTHERWISE} \end{cases}$$

LET Y BE THE POWER $x \rightarrow 2\Omega$

$Y = 2X^2$

FIND CUMMULA. DIST. FUNC. FOR Y

$$F_Y(t) = P(Y \le t)$$
$$= P(2X^2 \le t)$$
$$= P\left(-\sqrt{t/2} \le X \le \sqrt{t/2}\right)$$
$$= P\left(9 \le X \le \sqrt{t/2}\right)$$

10-22-70

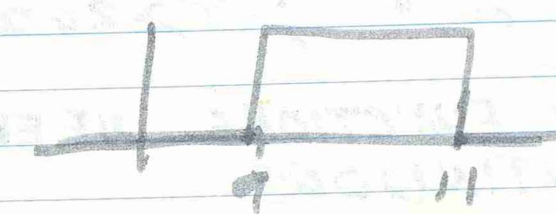
EXAM ON 2.8, CHAPT. 3 EXCEPT MOMENT GENERATING FUNCTIONS, INTERCORTILE RANGES, & PERCENTAGES

. . . .

(CONT) $Y = 2X^2$

$$f_x(x) = \begin{cases} \frac{1}{2} & 9 \le x \le 11 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$F_Y(t) = P(Y \le t)$$
$$= P(2X^2 \le t)$$
$$= P(X^2 \le t/2)$$
$$= P\left(\sqrt{t/2} \le X \le \sqrt{t/2}\right)$$



(CONT.)

$F_Y(t) = P(\sqrt{\tfrac{t}{2}} \le X \le 9) + P(9 \le X \le \sqrt{\tfrac{t}{2}})$
$\underset{0}{\shortparallel}$

$= P(9 \le X \le \sqrt{t/2})$

$= \begin{cases} 0 & ; \sqrt{t/2} \le 9 \\ \frac{1}{2}(\sqrt{\tfrac{t}{2}} - 9) & ; 9 \le \sqrt{t/2} \le 11 \\ 1 & ; \sqrt{t/2} \ge 11 \end{cases}$

$= \begin{cases} 0 & ; t \le 162 \\ \frac{1}{2}(\sqrt{t/2} - 9) & ; 162 \le t \le 242 \\ 1 & ; t \ge 242 \end{cases}$

= CUMMULATIVE DISTRIBUTION FUNCTION

FIND DENSITY FUNCTION BY DIFFERINTIATING:

$f_Y(t) = \frac{d}{dt} F_Y(t) = \begin{cases} 0 & ; t < 162 \\ \frac{1}{2\sqrt{2}} \frac{1}{2} t^{-\frac{1}{2}} & ; 162 \le t \le 242 \\ 0 & t > 262 \end{cases}$

$f_Y(y) = \begin{cases} 0 & ; Y < 162 \\ \frac{1}{4\sqrt{2}} y^{-\frac{1}{2}} & ; 162 \le Y \le 242 \\ 0 & ; Y > 262 \end{cases}$

DENSITY FUNCTIONS NEED NOT BE CONTINUOUS

CHECK THEM 3.5.2, pg. 102
(KNOW HOW TO DERIVE)

$f_X(x) = \begin{cases} \frac{1}{2} & ; 9 \le x \le 11 \\ 0 & ; \text{OTHERWISE} \end{cases}$

FROM THEM:

$f_Y(t) = \frac{1}{2\sqrt{at}} \left[f_X \sqrt{\tfrac{t}{a}} + f_X(-\sqrt{\tfrac{t}{a}}) \right]$ FOR $t \ge 0$

$(a=2)$ $f_X(-\sqrt{\tfrac{t}{a}}) = 0.0$

$f_X(\sqrt{t/2}) = \begin{cases} 1/2 & ; 9 \le \sqrt{t/2} \le 11 \\ 0 & ; \text{OTHERWISE} \end{cases}$

$f_Y(t) = \begin{cases} \frac{1}{2\sqrt{2}\sqrt{t}} \left(\frac{1}{2}\right) & \text{FOR } 9 \le \sqrt{t/2} \le 11 \\ 0 & \text{OTERWISE} \end{cases}$

$\therefore f_Y(t) = \frac{1}{4\sqrt{2}\sqrt{t}}$ FOR $162 \le t \le 242$
$= 0$ OTHERWISE

CHECKS WITH PREVIOUS ANSWER

THEM: LET $Y = a + bX$, WHERE X HAS THE MEAN μ_X AND VARIANCE σ_X^2. THEN
$\mu_Y = a + b\mu_X$ AND $\sigma_Y^2 = b^2 \sigma_X^2$

(CONT.)

PROOF: $\mu_Y = E(Y) = E(a+bX)$
$= a + bE(X)$
$= a + b\mu_x$

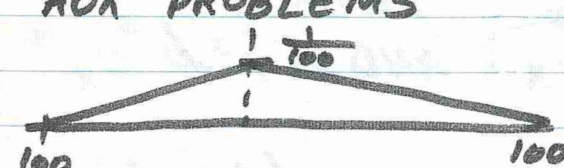
$\sigma_Y^2 = E\{(Y-\mu_Y)^2\}$
$= E\{[(a+bX) - (a+b\mu_x)]^2\}$
$= E\{(bX - b\mu_x)^2\}$
$= E\{b^2(X-\mu_x)^2\}$
$= b^2 E(X-\mu_x)^2$
$= b^2 \sigma_x^2$

DEFN: LET X BE A RANDOM VARIABE WITH MEAN μ_x AND VARIENCE σ_x^2.
THEN $Z = \frac{X-\mu_x}{\sigma_x}$ IS THE STANDARD FORM OF X

REMARK: Z IS DIMENTIONLESS. ALSO Z HAS MEAN $\mu_x = 0$ AND VARIANCE $\sigma_Z^2 = 1$

10-26-70

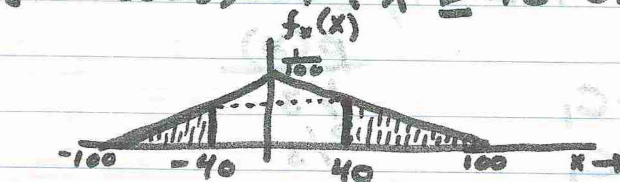
31)
AUX PROBLEMS



$f(x) = \frac{100+x}{10{,}000}$ $\quad -100 \le x \le 0$
$= \frac{100-x}{10{,}000}$ $\quad 0 \le x \le 100$
$= 0$ $\quad$ OTHERWISE

(CONT.)

(CONT.)
P(DAMAGE) = 1 - P(NO-DAMAGE)
ASSUME INDEPENDENCE
P(DAMAGE) = 1 - P(1 MISSES)P(2 MISSES)P(3 MISES)
P(1 MISSES) = $P(X \ge 40$ OR $X \le -40)$



P(1 MISSES) $= 2P(40 \le X \le 100)$
$= \frac{60}{10{,}000} \cdot 60 = .36$
$\therefore$ P(DAMAGE) $= 1 - (.36)^3$

Pg 90
11) BUY 1 TICKET; LET X BE YOUR GAIN

X	$p(x)$
2999	$\frac{1}{10{,}000}$
-1	$\frac{9999}{10{,}000}$

$\mu_x = E(X) = \frac{2999}{10{,}000} - \frac{9999}{10{,}000} = -.7$
$\therefore$ WOULD LOOSE 70¢ ON AVERAGE /DOLLAR

$\sigma_x^2 = E(X^2) - \mu_x^2$

IF 100 TICKETS BOUGHT

X	$p(x)$
3000 - 100	$1/100$
-100	$99/100$

$\mu_x = E(X) = 2900(\frac{1}{100}) - 100(\frac{99}{100})$
$= -70.$

AUX PROB

54) $f(t) = \begin{cases} k & 150 \le T \le 300 \\ 0 & \text{otherwise} \end{cases}$ $k = \frac{1}{150}$

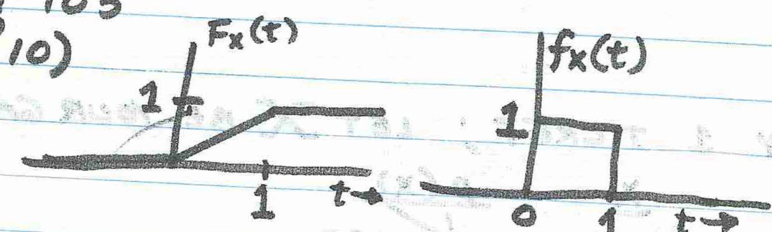
LET X = PROFIT/GALLON

	X	$p(x)$
IF $T \le 200$:	$C_2 - C_1$	$1/3$
IF $T \ge 200$:	$C_3 - C_1$	$2/3$

$$\therefore E(x) = \frac{C_2 - C_1}{3} + \frac{2(C_3 - C_1)}{3} = \frac{c^2 + 2C_3 - 3C_3}{3}$$

Pg 105

10)



$$F_Z(t) = P(Z \le t)$$
$$= P(X^2 \le t)$$
$$= P(-\sqrt{t} \le X \le +\sqrt{t})$$
$$= P(0 \le X \le \sqrt{t})$$
$$= \begin{cases} \sqrt{t} & 0 < \sqrt{t} < 1 \\ 1 & \sqrt{t} > 1 \end{cases}$$
$$= \begin{cases} 0 & t > 0 \\ \sqrt{t} & 0 < t < 1 \\ 1 & t > 1 \end{cases}$$

TEST

TOMORROW

10-28-70

WILL COVER CHAPT. 4 WITH 5.1-5.3

THE BINOMIAL RANDOM VARIABLE

DEFN: A BERNOULLI TRIAL IS AN EXPERIMENT WHICH HAS ONLY TWO POSSIBLE OUTCOMES, SAY SUCCESS OR FAILURE.

DEFN: A BINOMIAL EXPERIMENT IS A SET OF n INDEPENDENT BERNOULLI TRIALS WITH PROBABILITY OF SUCCESS p ON EACH TRIAL REMAINING THE SAME FROM TRIAL ~~OF~~ TO TRIAL

• • • • • SMITH IS A FAG • • • • •

DEFN: LET X BE THE NUMBER OF SUCCESSES IN A BINOMIAL EXPERIMENT OF n BERNOULLI TRIALS WITH PROBABILITY OF SUCCESS p ON EACH TRIAL. THEN X IS A BINOMIAL RANDOM VARIABLE WITH PARAMETERS n AND p, AND THE PROBABILITY FUNCTION OF X IS CALLED A BINOMIAL DISTRIBUTION

(CONT)

NOTE: IF $n=1$, THEN X IS ALSO CALLED A BERNOULLI RANDOM VARIABLE

THEM: THE BINOMIAL DISTRIBUTION IS GIVEN BY:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

FOR $x = 0, 1, 2, \ldots n$

PROOF: $P(x) = P(X = x)$
$= P$(x SUCCESSES IN n INDEPENDENT BOURNOULLI TRIALS WITH PROBABILITY p OF SUCCESS ON EACH TRIAL)

EX) LET $n = \cancel{3}4$; $x=2$

$p(SSFF) = p \cdot p \cdot (1-p) \cdot (1-p)$
$= p^2 (1-p)^{4-2}$

FOR (SSFF) IN ANY ORDER
$= \binom{4}{2} p^2 (1-p)^{4-2}$

(WHAT A LOUSY PROOF!)

$p(x) =$ (# OF ARRANGEMENTS OF x S's AND $n-x$ F's)
$\cdot\, p$(EACH ARRANGEMENT OF X's AND $(n-x)$ F's)

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

EX) TOSS 3 COINS; LET X BE THE # OF HEADS

$$p(x) = \frac{\binom{3}{x}}{2^3} \quad \text{(FOR } x = 0, 1, 2, 3\text{)}$$

$p = \frac{1}{2}$; $n = 3$

$$\therefore p(x) = \binom{3}{x} \left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{3-x} = \frac{\binom{3}{x}}{8}$$

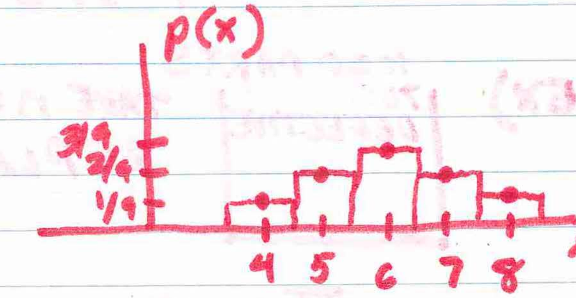
LET'S CHECK TO SEE IF ITS A PERFECTLY GOOD PROBABILITY FUNCTION

BINOMIAL THEM:
$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^n b^{n-x}$$

$$\sum_{\text{ALL } x} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$
$$= [p + (1-p)]^n$$
$$= 1^n = 1$$

IT'S PERFECTLY GOOD

x	$p(x)$
4	1/9
5	2/9
6	3/9
7	2/9
8	1/9



HISTOGRAM

10-30-70

TEST WENT ϕYEr

$\vdash * < \psi \psi \phi \uparrow \theta > \exists \forall \ni (N) \in \Leftrightarrow 0$

$\Leftrightarrow \forall \varepsilon > 0 \, \exists N = N(\varepsilon) \ni n > N \Rightarrow |a_n - a| < \varepsilon$

11-2-70

EX) TOSS 15 COINS. LET $\underline{X}$ = # OF HEADS.

FIND $P(\underline{X} = 11)$ $(n=15; p=\frac{1}{2})$

$P(\underline{X} = 11) = \binom{15}{11} \left(\frac{1}{2}\right)^{11} \left(\frac{1}{2}\right)^{4}$

USING TABLE ON pg 359.

FOR $t \in I$. $\Rightarrow$ GIVES $P(\leq t)$

$P(\underline{X}=11) = P(\underline{X} \leq 11) - P(\underline{X} \leq 10)$

$= .9824 - .9408$

$= .0416$

$P(\underline{X} > 11) = 1 - P(\underline{X} \leq 11)$

$= 1 - .9824 = .0176$

EX) 1000 PARTS, 700 DEFECTIVE. TAKE $n = 15$ WITH REPLACEMENT

LET $\underline{X}$ BE NUMBER OF DEFECTIVE PARTS IN THE SAMPLE OF 15

(CONT.)

FIND $P(\underline{X} = 11)$

$p = .7$

$P(\underline{X} = 11) = \binom{15}{11} (.7)^{11} (.3)^{4}$

TO USE TABLES, MUST REVERSE ROLES OF SUCCESS & FAILURE.

$\therefore$ LET $\underline{Y}$ = # OF NON-DEFECTIVES IN THE SAMPLE

$P(\underline{X} = 11) = P(\underline{Y} = 4)$

$p' = .3$;

$P(\underline{Y} = 4) = P(\underline{Y} \leq 4) - P(\underline{Y} \leq 3)$

$= .5155 - .2969$

$= .2186$

THEM: THE MEAN & VARIANCE OF A BINOMIAL DISTRIBUTION WITH PARAMETERS n & p ARE $\mu = np$ AND $\sigma_x^2 = npq$

PROOF: $E(x) = \mu_x$

$$= \sum_{\text{ALL } x} x f(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

(CONT.)

$$\mu_x = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

~~LET Y = X-1~~

$$\mu_x = \sum_{x=1}^{n} \frac{[n(n-1)!]}{(x-1)![(n-1)-(x-1)]!} p^{x-1} p \cdot (1-p)^{(n-1)-(x-1)}$$

LET Y = X-1; m = n-1

$$\mu_x = np \sum_{Y=0}^{m} \frac{m!}{Y!(m-Y)!} p^Y (1-p)^{m-Y}$$

$$\therefore \mu_x = E(x) = np$$

$$\sigma_x^2 = E(X^2) - \mu_x^2$$

$$E(X^2) = E[X(X-1)+X]$$

$$= E[X(X-1)] + E(X)$$

FIND $E[X(X-1)]$ THRU SIMILAR METHODS USED TO FIND μ_x

COMES OUT AS $\sigma_x^2 = npq$

DEFN: A GEOMETRIC RANDOM VARIABLE X WITH PARAMETER p IS THE NUMBER OF INDEPENDENT REPETED BERNOULLI TRIALS WITH PROBABILITY p OF SUCCESS ON EACH TRIAL REQUIRED TO OBTAIN THE FIRST SUCCESS. HOW 'BOUT THAT!

THEM: IF X IS A GEOMETRIC RANDOM VARIABLE WITH PARAMETER p, IT'S PROB. FUNCTION IS GIVEN BY:

$$p(x) = p(1-p)^{x-1} \text{ FOR } x = 1,2,3\dots$$

11-3-70

~~PROOF~~ OF ABOVE THEM:

$$p(x) = p(1-p)^{x-1}$$

PROOF: $p(x) = P(X=x) = P($IT TAKES x TRIALS TO GET THE FIRST SUCCESS$)$

$= P(x-1$ FAILURES IN A ROW AND 1 SUCCESS$)$

ALL OF THESE TRIALS ARE IND.

$$\therefore p(x) = \underbrace{(1-p)(1-p)\dots(1-p)}_{x-1} p$$

(CONT.)

$\therefore\ p(x) = p(1-p)^{x-1}$

IS THIS A GOOD PROB. FUNC.?

$$\sum_{\text{ALL } x} p(x) = \sum_{x=1}^{+\infty} p(1-p)^{x-1}$$
$$= p\sum_{x=1}^{+\infty}(1-p)^{x-1}$$
$$= p\left[1+(1-p)+(1-p)^2+(1-p)^3+\ldots\right]$$
$$= p\,\frac{1}{1-(1-p)} = 1$$

HOW ABOUT THAT!

MEAN AND VARIANCE OF $p(x)$

<u>THEM</u>: LET X BE A GEOMETRIC RANDOM VARIABLE WITH PARAMATER p. THEN $\mu_x = E(X) = 1/p$ AND $\sigma_x^2 = \frac{1-p}{p^2}$

PROOF:

$$\mu_x = E(x) = \sum_{x=1}^{\infty} x\,p(1-p)^{x-1}$$

THIS TAKES CARE OF THE GEOMETRIC DISTRIBUTION WHICH IS PERFECTLY GOOD.

<u>DEFN:</u> SUPPOSE A LOT CONTAINS M OBJECTS, W OF WHICH ARE CALLED SUCCESSES, AND $M-W$ OF WHICH ARE CALLED FAILURES. SUPPOSE WE TAKE A RANDOM SAMPLE OF n OF THESE OBJECTS <u>WITHOUT</u> REPLACEMENT FROM THIS LOT. LET X BE THE NUMBER OF SUCCESSES IN THE SAMPLE, THEN X IS A HYPER-GEOMETRIC RANDOM VARIABLE WITH PARAMATERS M, W, AND n, AND IT'S PROBABILITY FUNCTION IS CALLED A HYPER-GEOMETRIC DISTRIBUTION.

<u>THEM</u>: THE HYPER-GEOMETRIC DISTRIBUTION WITH PARAMATERS WITH PARAMATERS M, W, & n IS

$$p(x) = \frac{\binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}} \quad \text{FOR } x = 0,1,2,\ldots,n$$

(OVER FOR PROOF)

(CONT.)

PROOF: (IF $r > n$, THEN $\binom{n}{r} = 0$)

$$p(x) = P(X=x)$$

$$= \frac{\binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}}$$ ⇐ # OF COMBINATION W/O REGARD TO ORDER

CHECK TO SEE $\Sigma p = 1$

$$\sum_{\text{ALL } X} p(x) = \sum_{x=0}^{n} \frac{\binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}}$$

$$= \frac{\sum_{x=0}^{n} \binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}}$$

$$= \frac{\binom{W+M-W}{x+n-x}}{\binom{M}{n}} = \frac{\binom{M}{n}}{\binom{M}{n}} = 1$$

THEOREM: LET X BE A HYPERGEOMETRIC RANDOM VARIABLE WITH PARAMETERS $M, W, \& \, n$, THEN $\mu_x = n\,W/M$ AND $\sigma_x^2 = n\left(\frac{W}{M}\right)\left(1-\frac{W}{M}\right)\left(\frac{M-n}{M-1}\right)$

PROOF: DO AS AUX. PROBLEM FOR ~~μ_x~~ ONLY (WAAAA!)

HYPERGEOMETRIC RANDOM VARIABLE:
X IS THE NUMBER OF SUCCESSES OF n TAKEN WITHOUT REPLACEMENT WHEN THERE ARE W SUCCESSES IN LOT OF M

BINOMIAL RANDOM VARIABLE:
X IS THE NUMBER OF SUCCESSES OF n TAKEN WITH REPLACEMENT. IF THERE ARE W SUCCESSES IN LOT OF M, $p = \frac{W}{M}$

11-5-70

EX) $M = 10^8$ VOTERS
$n = 1000$ $W = 6 \times 10^7$
WITHOUT REPLACEMENT, ~~HAVE~~ BE HYPERGEOMETRIC

FIRST VOTOR $p = .6$
SECOND VOTER $p = \frac{59{,}999{,}999}{99{,}999{,}999} \cong .6$

FOR ALL PRACTICAL PURPOSES, THIS MAY BE TREATED AS A BI-NOMIAL RANDOM VARIABLE

THEM: LET X BE A HYPER-GEOMETRIC RANDOM VARIABLE WITH PARAMETERS M, W, AND n. THEN

$$\lim_{M\to\infty} \frac{\binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}} = \binom{n}{x} p^x q^{n-x}$$

$\ni \frac{W}{M} = p$ REMAINS CONSTANT

PROOF: GIVEN ON Pg. 53 OF PARZEN'S "MODERN PROBABILITY THEORY AND APPLICATIONS"

NUMERICAL EXAMPLE

M=100 W=60 n=10

$$P(X=5) = \frac{\binom{60}{5}\binom{40}{5}}{\binom{100}{10}}$$ WITHOUT REPLACEMENT

$$= .208$$

DO IT WITH REPLACEMENT

$p=.6$

$$p(X=5) = \binom{10}{5}(.6)^5(.4)^5 = .201$$

RULE OF THUMB: WE CAN USE BINOM. DIST. TO APPROX. HYPERGEO. DIST IFF $M \geq 10n$

PROB

DEFINITION: LET X BE A DISCRETE RANDOM VARIABLE AND LET α BE A POSITIVE CONSTANT. IF THE PROBABILITY FUNCTION FOR X IS

$$p(x) = \frac{e^{-\alpha}\alpha^x}{x!}; \; x=0,1,2,\ldots,$$

THEN X IS CALLED A POISON RANDOM VARIABLE, AND $p(x)$ IS A POISSON DISTRIBUTION, WITH PARAMETER α

$$\sum_{\text{ALL } x} p(x) = \sum_{x=0}^{\infty} \frac{e^{-\alpha}\alpha^x}{x!}$$

$$= e^{-\alpha}\sum_{x=0}^{\infty}\frac{\alpha^x}{x!}$$

$$= e^{-\alpha}\cdot e^{\alpha} = 1$$

THEOREM: LET X BE A POISSON RANDOM VARIABLE WITH PARAMATER α. THEN $\mu_x = \sigma_x^2 = \alpha$

(CONT)

PROOF:

$$\mu_x = \sum_{\text{ALL } x} x\,p(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\alpha}\alpha^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\alpha}\alpha^x}{(x-1)!}$$

$$= \alpha \sum_{x=1}^{\infty} \frac{e^{-\alpha}\alpha^{x-1}}{(x-1)!} \qquad \text{LET } y = x-1$$

$$= \alpha \sum_{y=0}^{\infty} \frac{e^{-\alpha}\alpha^y}{y!} = \alpha$$

$$\sigma_x^2 = E(x^2) - \mu_x^2$$

$$= E(x(x-1)) + E(x) - \mu_x^2$$

$$E[x(x-1)] = \sum_{\text{ALL } x} x(x-1) \frac{e^{-\alpha}\alpha^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\alpha}\alpha^x}{(x-2)!}$$

$$= \alpha^2 \sum_{x=2}^{\infty} \frac{e^{-\alpha}\alpha^{x-2}}{(x-2)!} = \alpha^2$$

$$\therefore \sigma_x^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$$

MUST KNOW HOW TO FIND MEAN & VARIENCE OF AT LEAST BINOMIAL & POISSON DISTRIBUTIONS, AS WELL AS THE FORMULA'S FOR EACH DISTRIBUTION

THEM: THE $\lim_{n \to \infty} \binom{n}{x} p^x q^{n-x} \ni np = \alpha$ REMAINS CONSTANT

$$= \frac{e^{-\alpha}\alpha^x}{x!}$$

11-6-70

PROOF OF ABOVE THEOREM

$$\binom{n}{x} p^x q^{n-x}$$

$$= \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad \alpha = np$$

$$\Rightarrow \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \left(\frac{\alpha}{n}\right)^x \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \lim_{n\to\infty} \frac{\alpha^x}{x!} \frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x} \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$= \frac{\alpha^x}{x!} \lim \frac{n}{n}\frac{n-2}{n}\frac{n-3}{n}\cdots\frac{n-x+1}{n} \frac{\left(1-\frac{\alpha}{n}\right)^{n-x}}{\left(1-\frac{x-n}{n}\right)}$$

$$= \frac{\alpha^x}{x!} \lim 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots \cdot \left(1-\frac{\alpha}{n}\right)^{n-x}$$

$$\lim \left(1+\frac{x}{n}\right)^n = e^x$$

$$= \frac{\alpha^x}{x!}(1)(1)(1)\cdots(1)\cdot e^{-\alpha}\cdot 1^{-x}$$

$$= \frac{\alpha^x}{x!} e^{-\alpha}$$

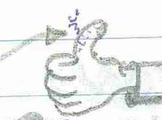
NUMERICAL EVIDENCE

$p = .04$ $n = 49$

x	$\binom{49}{x}(.04)^x(.96)^{49-x}$
0	.135
1	.276
2	.276
3	.180
4	.086
5	.032
6	.010
7	.003

x	POISON APRXOX $\alpha = np = 1.96$
0	.141
1	.276
2	.270
3	.176
4	.086
5	.034
6	.011
7	.003

COMPARE THE TWO TABLES.
APPROX. GOOD WITH p LITTLE AND n LARGE

RULE OF THUMB: 

WE CAN USE POISON DISTRIBUTION TO APPROXIMAT THE BINOMIAL DISTRI. IF $n \geq 20$ AND $p \leq .05$.
IF $n \geq 100$, WE CAN USE THE POISON APPROXIMATION IF $np \leq 10$

SUPPOSE $p(x,t)$ IS THE PROB. OF GETTING x SUCCESSES IN A TIME INTERVAL t, WHEN:

1) THE PROBABILITY OF EXACTLY ONE SUCCESS DURING A SMALL INCREMENT OF TIME Δt WITHIN THE TIME INTERVAL t IS APPROX $\lambda \Delta t$ WHERE λ IS POS. CONSTANT
2) THE PROBABILITY OF MORE THAN ONE SUCCESS IN THE INCREMENT OF TIME Δt IS FOR ALL PRACTICAL PURPOSES, 0.
3) THE OCCURANCE OF A SUCCESS IN THE INCREMENT IN Δt IS INDEPENDENT OF THE OCCURANCE OF A SUCCESS IN ANY OTHER TIME INCREMENT

DEFINITION: THE SEQUENCE OF SUCCESSES FORMED IN THE ABOVE MANNER IS CALLED A POISON PROCESS

THEOREM: LET X BE THE NUMBER OF SUCCESSES FORMED IN THIS WAY IN TIME INTERVAL t, THEN X IS A POISSON RANDOM VARIABLE WITH PARAMETER $\alpha = \lambda t$

PROOF (HEURISTIC):

$$P(X=x) \approx \binom{t/\Delta t}{x} (\lambda \Delta t)^x (1-\lambda\Delta t)^{\frac{t}{\Delta t}-x}$$

$$\approx \frac{e^{-\frac{\Delta t}{\Delta t}\lambda \Delta t} \frac{t}{\Delta t}(\lambda \Delta x)^x}{x!}$$

$$= \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

11-9-70

$p(x,t)$ = PROB. OF x SUCCESSES IN TIME t

$$p(x,t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

$\ni \lambda t = \alpha = \mu_x \Rightarrow \lambda$ = MEAN COUNT RATE/TIME

USING TABLES:

$$P(x \le 1.5) = F(1.5) = \sum_0^{[1.5]=1} \frac{\alpha^x e^{-\alpha}}{x!}$$

Pg 129 #10

$t = 1$ COOKIE

X IS THE # OF CHIPS IN $t = 1$ COOKIES

λ = AVERAGE # OF CHIPS/COOKIE = 3

$\alpha = \lambda t = 3 \cdot 1 = 3$

$P(X=0) = \frac{e^{-3}\,3^0}{0!} = P(X \le 0) = .050$

$P(X=3) = P(X \le 3) - P(X \le 3)$

$= .647 - .423$

$= .224$

$P(X=1) = P(X \le 1) - P(X \le 0)$

$= .199 - .050$

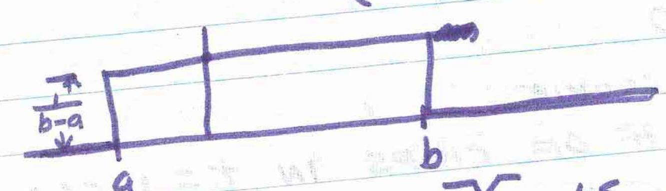
$= .149$

Y = # OF COOKIES IN 1000 WITH 1 CHIP

$E(Y) = 1000(.149) = 149$

DEF: THE CONTINUOUS RANDOM VARIABLE X IS A UNIFORM (OR RECTANGULAR) IFF ITS DENSITY FUNCTION (THE UNIFORM OR RECTANGULAR DISTRIBUTION), IS OF THE FORM:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{FOR } a \le x \le b \\ 0 & \text{OTHERWISE} \end{cases}$$



THEOREM: IF X IS A UNIFORM RANDOM VARIABLE WITH THE ABOVE DENSITY FUNCTION, THEN:

$\mu_x = \frac{b-a}{2}$ AND $\sigma_x^2 = \frac{(b-a)^2}{12}$

DEF: THE CONTINUOUS RANDOM VARIABLE X IS EXPODENTIAL WITH PARAMETERS $\theta > 0$ IF ITS DENSITY FUNCTION (THE EXPODENTIAL DISTRIBUTION) IS OF THE FORM:

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & x \ge 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

THEM: THE EXPODENTIAL RANDOM VARIABLE, WITH PARAMETER θ HAS MEAN $\mu_x = \theta$ AND VARIANCE $\sigma_x^2 = \theta^2$

CHECK THE EXPO. INTEGRAL

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{+\infty} \frac{1}{\theta} e^{-x/\theta}\,d\theta = -e^{-x/\theta} = 0 - e^{-0} = 1$$

PROOF OF ABOVE THEOREM

$$\mu_x = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$= \frac{1}{\theta}\int_{-0}^{\infty} x e^{-x/\theta}\,dx$$

$$= \frac{1}{\theta} e^{-\frac{x}{\theta}}\left[\frac{x}{-1/\theta} - \frac{1}{(-1/\theta)^2}\right]_0^{\infty}$$

$$= 0 - \frac{1}{\theta}\left[-\theta^2\right] = \theta$$

$$E(X^2) = \frac{1}{\theta}\int_0^{\infty} x^2 e^{-\frac{x}{\theta}}\,d\theta$$

INTEGRATE, PLUG AND CHUG. DO AS AN AUXILERY PROBLEM (SNICKER, SNIFF, SNICKER)

THEOREM: LET T BE THE TIME NECCESARY TO GET THE FIRST SUCCESS IN A POISSON PROCESS, WITH MEAN RATE PER UNIT RATE λ, THEN T IS AN EXPODENTIAL RANDOM VARIABLE WITH PARAMETER $\theta = 1/\lambda$

PROOF: $F_T(t) = P(T \le t)$

$= 1 - P(T \ge t)$

$= 1 - \frac{e^{-\lambda t}(\lambda t)^0}{0!} = -e^{-\lambda t} + 1$

DENSITY FUNCTION:

$\frac{d}{dt} F_T(t) = f(t) = \lambda e^{-\lambda t}$

11-10-70

EXAM ON CHAPT 4 THRU NORMAL DIST.

PROBLEM SESSION THURS 4-6 PM. IN B1A

$f(x) = \frac{1}{\theta} e^{-x/\theta}$ $x \ge 0$; $\theta > 0$

$= 0$ OTHERWISE

$\mu_x = \theta$ $\sigma_x^2 = \theta^2$

$F_T(t) = P(T \le t) = 1 - e^{-\lambda t}$; $t \ge 0$

$\frac{dF_T(t)}{dt} = f_T(t) = \lambda e^{-\lambda t}$ $t > 0$

~~XMEK~~ $\lambda = \frac{\text{MEAN RATE OF SUCCESSES}}{\text{UNIT TIME}}$

$E(T) = 1/\lambda$ = MEAN TIME FOR 1ST SUCCESS

NORMAL DISTRIBUTION:

DEFN: LET X BE A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION OF THE FORM:

$f(x) = \frac{1}{2\pi} \frac{1}{b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$ $-\infty < x < \infty$

WHERE a IS ANY REAL # AND $b > 0$. THEN X IS CALLED A NORMAL RANDOM VARIABLE, AND f(x) IS CALLED A NORMAL DISTRIBUTION

PROB

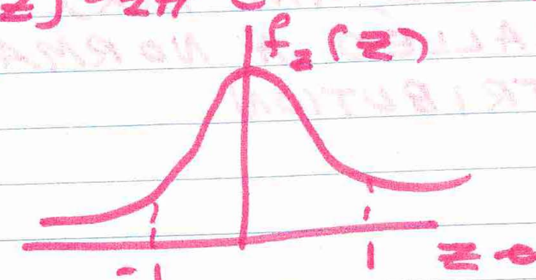
<u>THEM:</u> IF X IS A NORMAL RANDOM VARIABLE WITH THE ABOVE DENSITY FUNCTION, $\mu_X = E(X) = a$ AND $\sigma_x = b$
(PROOF TOMORROW → ON TEST)

<u>REMARK:</u> THUS WE ORDINARILY WRITE $f(x) = \frac{1}{2\pi} \frac{1}{\sigma} \text{EXP}\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$
KNOW FOR TEST REAL WELL

<u>REMARK:</u> WE OFTEN USE NOTATION $N(\mu, \sigma^2)$ TO REPRESENT A NORMAL DISTRIBUTION WITH MEAN μ AND VARIENCE σ^2

<u>DEFN:</u> THE $N(0,1)$ RANDOM VARIABLE IS CALLED THE STANDARDIZED NORMAL RANDOM VARIABLE, & IS DENOTED BY Z

$$f_z(z) = \frac{1}{2\pi} e^{-\frac{1}{2}z^2} \quad -\infty < z < \infty$$



$$P(0<z<1) = \int_0^1 \frac{1}{2\pi} e^{-\frac{1}{2}z^2} dx$$

USE TABLES

$$N_z(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$P(0<z<1) = P(z<1) - P(z<0)$$
$$= .8413 - .5 = .3413$$

<u>THEM:</u> LET X BE A $N(\mu, \sigma^2)$ RANDOM VARIABLE, THEN $Z = \frac{X-\mu}{\sigma}$ IS A $N(0,1)$ RANDOM VARIABLE

– LET X BE A $N(70, 25)$

$$P(65<X<80) = P\left(\frac{65-70}{5} < \frac{X-70}{5} < \frac{80-70}{5}\right)$$

STANDARDIZED NOR DIST

$$= P(-1<Z<2)$$

(WHEW!)

11-12-70

EXAM TOMMORROW ON CHAPT. 4

HELP SESSION IN A-241

AUX. PROB. #74

$\mu = .397$; $\sigma = .005$

FIND $P(X > .400)$

SHOULD PUT IN STANDARD FORM

$P(X > .400) = P\left(\frac{X-.397}{.005} > \frac{.400-.397}{.005}\right)$

$= P(Z > 3/5) = P(Z > .6)$

$= 1.0 - P(Z < .6)$

$= 1 - .7257 = .2743$

PG 137, # 8

$\lambda = 50/HR$

LET PURCHASE TIME = 0

$\theta = 1/\lambda$

a) $P(T \geq 2) = \int_{\frac{1}{30}}^{\infty} (50)\, e^{-50t}\, dt$

$= -e^{-50t}\Big]_{1/30}^{\infty}$

$= e^{-5/3}$

$= .1882$

b) WOULD BE THE SAME THING

~~$P(T>7) = P(T>5)$~~

$P(T>7 \mid T>5) = P(T>2)$

AUX PROBS: #64

$p_Y(y) = P(Y=y) = \begin{cases} \frac{e^{-\lambda}\lambda^y}{y!} & \text{FOR } Y=1-29 \\ \phi & \text{FOR } Y=30 \end{cases}$

$f_X(x) = \frac{e^{-x}\lambda^x}{x!}$ FOR $X = 0, 1, 2 \ldots$

$\therefore P(Y=30) = P(X \geq 30) = \phi$

$\Rightarrow \phi = \sum_{X=30}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$

$= 1 - \sum_{X=0}^{29} \frac{e^{-\lambda}\lambda^x}{x!}$

PG. 120 #5

10 BALLS ; 1 BLACK

Z	p(Z)
1	1/10
2	$\frac{9}{10}\cdot\frac{1}{9} = \frac{1}{10}$
3	$\frac{9}{10}\cdot\frac{8}{9}\cdot\frac{1}{8} = \frac{1}{10}$
4	$\frac{9}{10}\cdot\frac{8}{9}\cdot\frac{7}{8}\cdot\frac{1}{7} = \frac{1}{10}$
⋮	
10	$\frac{1}{10}$

$\therefore p(z) = \frac{1}{10}$ FOR $Z = 1, 2, 3, \ldots, 10$

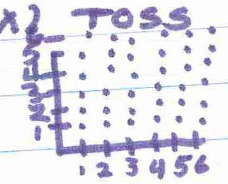
$E(Z) = \sum_{Z=1}^{10} Z \frac{1}{10} = 5.5$

11-16-70

Pg 150 1-7 Pg 158-9 1-8 Pg 169-70 1,2,3,... 7,9,10, 12,13
+ ALL REMAINING AUX PROB. OMMIT REGRESSION

DEFN: LET X AND Y BE 2 DISCRETE RANDOM VARIABLES DEFINED ON SAMPLE SPACE S. THEN THE JOINT PROB. FUNC. FOR X AND Y IS DEFINED TO BE
$p(X=x \text{ AND } Y=y)\ \forall\ (x,y)$ PAIR.

EX) TOSS 2 DICE. LET X BE # OF 1'S, AND Y = # OF 2'S ON TOSS OF TWO DICE



X	p(x)
0	25/36
1	10/36
2	1/36

Y	p(y)
0	25/36
1	10/36
2	1/36

p(x,y)	X = 0	1	2
Y = 0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0

$p(0,2) = P(X=0 \,\&\, Y=2)$

$\sum_{ALL\ X} p(x,y) = 1$

REMARK: IF $p(x,y)$ IS A GOOD PROB FUNC, THEN $p(x,y) \in [0,1]\ \forall\ (x,y)$ & $\sum\sum_{ALL\ XY} p(x,y) = 1$

11-19-20

$\sigma_{XY} = E(XY) - \mu_X \mu_Y$

EX)

Y \ X	0	1	2	$p_Y(y)$
0	16/36	8/36	1/36	25/36
1	8/36	2/36	0	10/36
2	1/36	0	0	1/36 (CONT.)
	25/36	10/36	1/36	= 1

$$E(XY) = \sum_{ALL}\sum_{X} x\,y\,p(x,y)$$

$$E(XY) = 0\cdot0\cdot\tfrac{16}{36} + 0\cdot1\cdot\tfrac{8}{36} + 0\cdot2\cdot\tfrac{1}{36} + 1\cdot0\cdot 8/36 + 1\cdot1\cdot 2/36 + 1\cdot2\cdot0 + 2\cdot0\cdot 1/36 + 2\cdot1\cdot0 + 2\cdot2\cdot0 = 2/36 = 1/18$$

$$\mu_x = \sum_{ALL\ X} x\,p_x(x) = 1/3$$

$$\mu_y = \sum_{ALL\ Y} y\,p_x(x) = 1/3$$

$$\therefore \sigma_{XY} = \tfrac{1}{18} - \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) = -\tfrac{1}{18}$$

COMPUTE CORRELATION COEFFITIENT

$$\rho = \sigma_{XY}/\sigma_x\sigma_y$$

$$\sigma_x^2 = E(X^2) - \mu_x^2$$

$$E(X^2) = \sum_{ALL\ X} x^2 p(x) = 0^2\tfrac{25}{36} + 1^2\tfrac{10}{36} + 2^2\tfrac{1}{36} = 7/18$$

$$\therefore \sigma_x^2 = 7/18 - \left(\tfrac{1}{3}\right)^3 = 5/18$$

$$\Rightarrow \sigma_x = \sqrt{5/18}$$

$$\sigma_y^2 = E(Y^2) + \mu_y^2 = \sum_{ALL\ Y} y^2 p(y) - \mu_y^2$$

IN THIS CASE, $\sigma_x = \sigma_y = \sqrt{5/18}$

$$\Rightarrow \rho = \frac{\sigma_{XY}}{\sigma_x\sigma_y} = \frac{-1/18}{5/18} = -1/5$$

THEM: $-1 \le \rho \le +1$. IF $Y = aX + b$ (Y IS LINEAR FUNC. OF X), THEN $\rho^2 = +1$

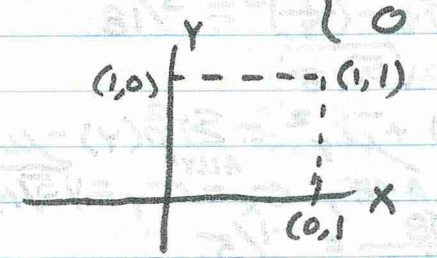
PROB

<u>DEF</u>: LET X AND Y BE DISCRETE (CONTINUOUS) RANDOM VARIABLES WITH JOINT PROBABILITY FUNCTION $p(x,y)$ (JOINT DENSITY FUNCTION $f(x,y)$) AND MARGINAL PROBABILITY FUNCTION $p_x(x)$ \& $p_Y(y)$ (MARGINAL DENSITY FUNCTION $f_x(x)$ \& $f_Y(y)$) THEN X \& Y ARE INDEPENDENT IFF: $p(x,y) = p_x(x)\,p_Y(y)\ \forall\ x,y$ $\left(f(x,y) = f_x(x) f_Y(y)\ \forall\ Y\right)$

● <u>THEM</u>: IF X \& Y ARE INDEPENDENT RANDOM VARIABLE THEN $\sigma_{xy} = 0$

<u>REMARK</u>: CONVERSE IS NOT TRUE

EX) $f(x,y) = \begin{cases} 4xy & \text{FOR } 0<x<1 \ \& \ 0<y<1 \\ 0 & \text{OTHERWISE} \end{cases}$



FIND MARGINAL DENSITY FUNCS.

$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

IF $0<x<1$, THEN $f_x(x) = \int_0^1 4xy\,dy$

$= \frac{4xy^2}{2}\Big|_0^1 = 2X$

$\Rightarrow \cancel{f_x}(x) = \begin{cases} 2X & 0<1<1 \\ 0 & \text{OTHERWISE} \end{cases}$

IF $0<y<1$, $f_Y(y) = \int_0^1 4xy\,dx = 2Y$

$\therefore f_Y(y) \begin{cases} = 2Y & 0<Y<1 \\ = 0 & \text{OTHERWISE} \end{cases}$

$f_x(x) f_y(Y) = (2X)(2Y) = 4XY = f(x,y)$

$\Rightarrow$ THESE TWO ARE IND.

$\sigma_{XY} = E(XY) - \mu_X \mu_Y$

$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$

$= \int_0^1\int_0^1 xy\ 4xy\,dx\,dy$

$= \int_0^1\int_0^1 4x^2y^2\,dx\,dy = 4/9$

$\mu_x = \int_0^1 x\,2x\,dx = 2/3$

$\mu_Y = \int_0^1 y\,2y\,dy = 2/3$

$\sigma_{xy} = \frac{4}{9} - \frac{2}{3}\frac{2}{3} = 0 \Rightarrow \rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = 0$

FINAL:

1/3 OPEN BOOK

BRING SLIDE RULES

1/6 ON 5.1 – 5.3

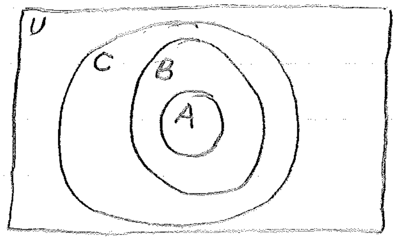
KNOW:

- GET JOINT P. FUNC. FROM S.S.
- GIVEN JOINT P. OR D. FUNCTION, FIND MARGINAL P. OR D. FUNC.
- FIND PROBABILITIES
- COMPUTE COVARIENCE OF RANDOM CORR. BETW 2 D. OR P. FUNCTION
- SHOW J.P.F. OR J.D.F. IS GOOD
- DEF. OF IND. OF RAN. VAR. AND SHOW

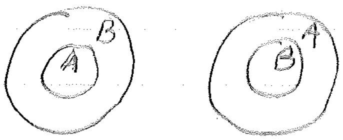
PROB

Pg 3-4 9-12-70

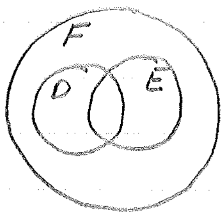
1)



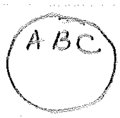
2)



3)



4)



5) a) F h) T
b) F i) T
c) T j) T
d) T k) F
e) T l) F
f) T m) F
g) F n) F

6) a) T e) F
b) F f) T
c) F g) T
d) T

7) YES, NO, NO

Pg 10-11 9-12-70

1) PEOPLE IN THE WORLD; PEOPLE LIVING IN THAT COUNTRY

2) $\bar{A} = \{4, 5, 6, 7 \ldots n\}$ $\bar{B} = \{1, 4, 5, 6, 7 \ldots (n-1)\}$

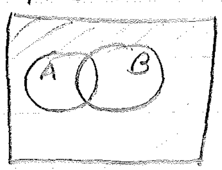
$\bar{A} \cup \bar{B} = \{1, 4, 5, 6 \ldots, n\}$

$A \cap B = \{2, 3\} \Rightarrow \overline{A \cap B} = \{1, 4, 5, 6 \ldots n\} = \bar{A} \cup \bar{B}$

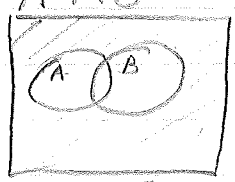
3) NO; YES $\Rightarrow \emptyset$

4) 

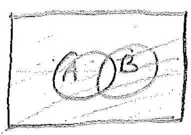
5) $\overline{A \cup B}$



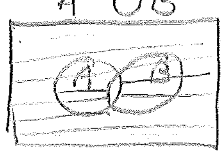
$\bar{A} \cap \bar{B}$



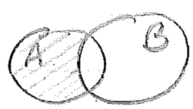
6) $\overline{A \cap B}$



$\bar{A} \cup \bar{B}$

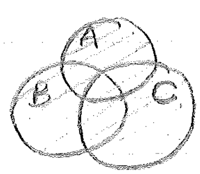


7) a)

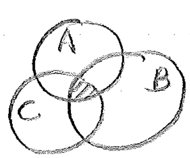


b)

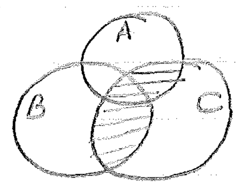
c)



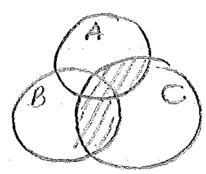
d)



e)



f)



8) a) $\{(1,2)(1,1)(2,2)(2,1)\}$

b) $\{(1,10)(1,12)(2,10)(2,12)$ ETC

9) $A = B$

10) NO

11) ?

Pg 13-14 9-15-70

1) DOMAIN = $\{X \mid X$ INDIVIDUALS IN FAMILY$\}$

RANGE: $\{Y \mid Y$ IS CORRESPONDING AGE OF INDIVIDUALS$\}$

$f(\omega) = \{(BOB, 20)\{MOM, 42\}(DAD, 41)(RAY, 6)\}$

2) 2 TO 12 $\Rightarrow \{2, 3, 4, 5 \ldots 12\}$

3) -1 TO 2 $\Rightarrow \{-1, 0, 1, 2\}$

4) $A_7 = \{(1,6), (2,5), (3,4), (6,1), (5,2), (4,3)\}$

$A_3 = \{(1,2), (2,1)\}$

$A_{10} = \{(4,6), (5,5), (6,4)\}$

5) $n(A) \Rightarrow 2^K$ ELEMENTS IN $\tilde{F}$

$n(A) = \{A \mid A$ IS A SUBSET OF $S\}$

6)

Pg 22·23

1) $S = \{1, 2, 3, 4, 5 \ldots 10\}$

2) $S = \{(X_1, X_2) \mid X_1 = 1, 2 \ldots 10 ; X_2 = 1, 2 \ldots 10\}$

3) $S = \{(X_1, X_2) \mid X_i = 1, 2, 3 \cdots 10 ; X_1 \neq X_2\}$

4) $A = \{1, 2, 3, 4\}$; $B = \{6, 7 \ldots 10\}$

5) $C = \{(X_1, X_2) \mid X_1 = 1, 2, 3, 4 \quad X_2 = 1, 2, 3 \ldots 10\}$

$D = \{X_1, X_2\} \mid X_1 = 1, 2, 3, 4 \ldots 10 ; \quad X_2 = 1, 2, 3, 4\}$

$E = \{X_1, X_2\} \mid X_i = 1, 2, 3, 4 ; i = 1, 2$

YES

6) $S = \{X_1, X_2 \mid X_i = a, b, c, d \; ; i = 1, 2\}$

7) $S_A = \{(X_1, X_2) \mid X_1 = (\text{BALD, BR. HAIR, BLACK HAIR}) ; X_2 = \{\text{BLUE, BROWN}\}\}$

$A = \{(X_1, X_2) \mid X_1 = \text{BALD} ; X_2 = (\text{BLUE, BROWN})\}$

$B = \{(X_1, X_2) \mid X_1 = \text{BALD, BR. HAIR, BL. HAIR}) ; X_2 = \{\text{BLUE})\}$

$C = \{(X_1, X_2) \mid X_1 = \text{BR HAIR} ; X_2 = (\text{BROWN})\}$

8) $S = \{X_1, X_2 \mid X_i = M, S, T \; ; i = 1, 2, \; X_1 \neq X_2\}$

$A = \{X_1, X_2 \mid X_1 = M \; ; X_2 = S, T\}$

$B = \{X_1, X_2 \mid X_1 = S, T ; X_2 = M\}$

$C = \{X_1, X_2 \mid X_i = T, S \; ; i = 1, 2 \; ; X_1 \neq X_2\}$

9) $S = \{(X_1 X_2, X_3, X_4 ; X_i = 0, 1, 2, 3 ; i = 1, 2, 3 4\}$

$A = \{0.0, 0, 0)$

$B = \{0, 3, 4)$

$C = \{A_1, A_2, A_3 \mid A_i = 0, 1, 2, 3 \text{ AND } A_1 \neq A_2 \neq A_3\}$

$D = \phi$

10) $S = \{X_1, X_2, X_3, X_4 \mid X_i = 0, 1, 2, 3, 4 ; i = 1, 2, 3$

$A = \{4, 00\}$

$B = \{2, 2, 0\}$

$C = \{1, 1, 2\}$

Pg 31-32; 9-21-70

1) $\frac{1}{2}$

2) $\frac{1}{2}$; $\frac{1}{4}$; $\frac{4}{52}$; $\frac{1}{52}$

3) $\frac{1}{5}$; $\frac{1}{5}$; $\cancel{\frac{1}{25}}$

4)

SUM	2	3	4	5	6	7	8	9	10	11	12
PROB	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36
	1	2	3	4	5	6					

5)

C_1	C_2	C_3	
1	1	1	= 3
1	1	2	= 4
1	2	1	= 4
1	2	2	= 5
2	1	1	= 4
2	1	2	= 5
2	2	1	= 5
2	2	2	= 6

$P(3) = 1/8$
$P(4) = 3/8$
$P(5) = 3/8$
$P(6) = 1/8$

6) P(MAN) = 18/40 = 9/20
P(GR CT) = 6/40 = 3/20
P(BR CT) = 7/40
P(NOR) = 15/40 = 3/8
P(GER) = 17/40
P(GER W/ GR CT) = 1/10

7) $1 = X$; $2 = 2X$; $3 = 3X$; $4 = 4X$ ...
$1X + 2X + 3X + 4X + 5X + 6X = 1$
$X = 1/21$
$P(2, 4, 6) = 12/21$
$P(5, 6) = 11/21$

Pg 37-8 9-23-70

1) $P^n_r = P^3_3 = \frac{3!}{0!} = 6$

2) $P^3_3 = 6$

3) $P^6_6 = 6! = 30 \cdot 24 = 720$

4) $27 (= 3^3)$

5) $2^4 = 16$

6) $P^4_3 = \frac{4!}{1!} = 24$

7) $2^3 = 8$

8) NO

9) $P^{28}_5 = \frac{28!}{23!}$

10) $P^8_2 = \frac{8!}{6!} = 56$

11) $P^{10}_3 = \frac{10!}{3!} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

12) $P^{20}_5 = \frac{20!}{15!} = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

13) P^{15}_2

14) $P^{15}_3 \; ; \; P^{15}_K$

15) $9!/(2!\,3!\,4!)$

16) 1

Pg 27-28

1) a) $P(C) = \frac{1}{3}$

b) $P(A \cup B) = \frac{2}{3}$

c) $P(\bar{A}) = \frac{2}{3}$

d) $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$

e) $P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$
$= 2/3 + 2/3 - 1/3 = 1$

f) $P(B \cup C) = 2/3$

2) a) $P(C) = 1 - \frac{7}{10} = 3/10$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= \frac{1}{2} + \frac{1}{5} - 0 = 7/10$

c) $P(\bar{A}) = P(B) + P(C) = 1/5 + 3/10 = 1/2$

d) 1

e) $P(B \cup C) = P(B) + P(C) - P(B \cap C)$
$= 1/5 + 3/10 = 1/2$

3) a) 0 d) 0
b) 1 c) 1
c) 0 d) 0 NO

5) PROVE $P(A) \leq P(B)$ IF $A \subset B$

$A \subset B \Rightarrow A \cup B = A$ & $A \cap B = B$

LET A = SAMPLE SPACE

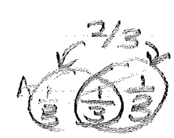
$\Rightarrow P(A) = 1$; $P(B) \geq 1$

$\therefore P(A) \geq P(B)$

7) a) $\frac{1}{2}$

b) $\frac{1}{2}$

c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$P(A \cap B) = \frac{1}{2} + \frac{1}{2} - \frac{2}{3} = 1/3$



d) $\frac{1}{3}$

e) $2/3$

f) $1/3$

Pg 42-3 10-4-70

1) $\binom{30}{12}$

2) $6!$

3) $2 \cdot 5!$

4) $\frac{20!}{15!\,5!} = \binom{20}{15}$

$\frac{20!}{10!\,10!} = \binom{20}{10}$

5) a) $\binom{8}{3}/\binom{10}{3}$

b) $\binom{8}{1}/\binom{10}{3}$

6) a) $\frac{\binom{99}{1}}{\binom{100}{2}}$

b) $\frac{1}{\binom{100}{2}}$

7) $\left(\frac{1}{2}\right)^{30}$

$2\left(\frac{1}{2}\right)^{30}$

8) a) $\frac{3}{6} = \frac{1}{2}$

b) $\frac{2}{6} = \frac{1}{3}$

9) a) $\left(\frac{1}{2}\right)^5$

b) $6\left(\frac{1}{2}\right)^5$

10) a) $\frac{\binom{4}{2}\binom{4}{2}\binom{13}{2}\binom{4,4}{1}}{\binom{52}{5}}$

b) $\frac{\binom{13}{1}\binom{4}{3}\binom{4}{2}\binom{13}{2}}{\binom{52}{5}}$

c) $\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$

d) $\frac{\binom{10}{1}\binom{4}{1}^5}{\binom{52}{5}}$

~~11)~~ $\frac{\binom{12}{1}}{\binom{12}{2}}$

~~12)~~ $\frac{}{2^{10}}$

13) a) $\frac{9}{\binom{10}{2}} = \frac{9}{\frac{10 \cdot 9}{2}} = \frac{1}{5}$

b) $\frac{1}{\binom{10}{2}} = \frac{2}{10 \cdot 9} = \frac{1}{45}$

14) a) $\frac{1}{\binom{6}{3}} = \frac{1}{\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}} = \frac{1}{20}$

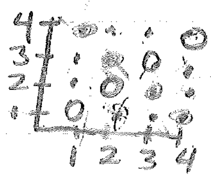
b) $\frac{2}{\binom{6}{3}} = \frac{1}{10}$

~~c)~~ $\frac{}{\binom{6}{3}} =$

Pp. 48-49   10-6-70

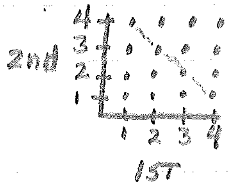
1) a) A = SUM IS 5 $B_i = i$ ON FIRST BALL

$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)} =$

2 

$A \cap B_i = \frac{\frac{1}{12}}{\frac{3}{12}} = \frac{1}{3}$

b) $P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} = \frac{\frac{1}{12}}{\frac{4}{12}} = \frac{1}{4}$

2) 

$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)}$

$= \frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}$

$P(B/A) = \frac{P(A \cap B_i)}{P(A)} = \frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}$

3) $\frac{1}{2}$

4) $\frac{1}{2}, \frac{1}{2}$

~~5)~~ U_1 = event urn 1 chosen

U_2 = " " 2 "

R = red ball is chosen from chosen urn

B = blue " " " " " "

$U = U_1 \cup U_2$

$P(R/U) = \frac{P(R \cap U)}{P(U)} = \frac{P(R \cap (U_1 \cup U_2))}{P(U_1 \cup U_2)}$

$= P$

5) a) $P_{BU_1} = \frac{2}{3}$ $P_{BU_2} = \frac{1}{6}$

$\frac{5}{6} \cdot \frac{1}{2} = 5/12$

b) $P_{RU_1} = \frac{1}{3}$ $P_{RU_2} = 5/6$

$7/6 \cdot \frac{1}{2} = 7/12$

AUXILERY PROBLEMS

1) $\binom{5}{2} = \frac{5!}{3!\,2!} = \frac{5\cdot 4}{2} = 10$

3) a) A = both balls are white

A$_1$ = first ball is white ; $P(A_1) = \frac{4}{12} = \frac{1}{3}$

A$_2$ = second " " " ; $P(A_2) = \frac{6}{18} = \frac{1}{3}$

$P(A) = P(A_1)P(A_2) = 1/9$

b) $P(R_1) = 3/12$; $P(R_2) = 5/18 \Rightarrow P(R) = 1/4 \;\; 5/18 = \frac{5}{72}$

$P(B_1) = 5/12$; $P(B_2) = 7/18 \Rightarrow P(B) = \frac{35}{216}$

$P(W) = 1/9$

$P(\text{SAME COLOR}) = \frac{5}{72} + \frac{35}{216} + \frac{1}{9}$

4) $P(2) = P(12) = 1/36$

$P(3) = P(11) = 2/36$

$P(4) = P(10) = 3/36$

$P(5) = P(9) = 4/36$

$P(6) = P(8) = 5/36$

$P(7) = 6/36$

5) 365^4 POSSIBILITIES

A = event 2 have same birthday

$P(\bar{A}) = 365\cdot 364\cdot 363\cdot 362$

$P(A) = 1 - {}_{365}P_4 / 365^4$

6) a) $n(S) = \binom{30}{3}$

$P(\text{NO DEFECTIVES}) = \frac{25}{30}\cdot\frac{24}{29}\cdot\frac{23}{28}$

b) $P(\text{3 DEFECTIVES}) = \frac{5}{30}\,\frac{4}{29}\cdot\frac{3}{28} \quad \left(= \frac{\binom{25}{3}}{\binom{30}{3}} \right)$

c) A = (1 DEF AND 2 NON-DEF)

$P(A) = \frac{\binom{5}{1}\binom{25}{2}}{\binom{30}{3}}$

12) a)

	B_1	B_2	W_1	W_2	W_3	R_1	R_2
B_1	O	✓	-	-	-	-	-
B_2	✓	O	-	-	-	-	-
W_1	·	·	O	✓	✓	-	-
W_2	·	·	✓	O	✓	-	-
W_3	·	·	✓	✓	O	-	-
R_1	-	-	-	-	-	O	✓
R_2	-	-	-	-	-	✓	O

$P(\text{SAME COLOR}) = \frac{10}{42}$

b) $P(\text{SAME}) = \frac{2}{7}\cdot\frac{1}{6} + \frac{3}{7}\cdot\frac{2}{6} + \frac{2}{7}\cdot\frac{1}{6}$

$$\frac{\binom{2}{2}+\binom{3}{2}+\binom{2}{2}}{\binom{7}{2}} = \frac{10}{42}$$

Pg 49 10-4-70 (CONT)

9) A = EVENT BROWN BALL IS INITIALLY DRAWN

B = " " " " FINALLY DRAWN

$P(A) = \frac{5}{7}$ $P(\bar{A}) = \frac{2}{7}$

$P(B/A) = 7/9$ $P(B/\bar{A}) = 5/6$

$$B = (B \cap A) \cup (B \cap \bar{A})$$

$$(B \cap A) \cap (B \cap \bar{A}) = \phi$$

$$\Rightarrow P(B) = P(B \cap A) + P(B \cap \bar{A})$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(B) = P(B/A)\,P(A) + P(B/\bar{A})\,P(\bar{A})$$

$$= \left(\tfrac{7}{9}\right)\left(\tfrac{5}{7}\right) + \left(\tfrac{5}{6}\right)\left(\tfrac{2}{7}\right)$$

$$= \frac{5}{9} + \frac{5}{21}$$

$$= \frac{45 + 105}{189} = \frac{150}{189} = \frac{50}{63}$$

10) $P(A/B)$

$$P(A/B) = \frac{P(B/A)\,P(A)}{\Sigma\, P(A)_i\, P(B/A_i)}$$

$$= \frac{\left(\frac{7}{9}\right)\left(\frac{5}{7}\right)}{\left(\frac{7}{9}\right)\left(\frac{5}{7}\right) + \left(\frac{2}{7}\right)\left(\frac{5}{6}\right)} = \frac{\frac{5}{9}}{\frac{5}{9} + \frac{5}{21}} = \frac{\frac{5}{9}}{\frac{50}{63}} = \frac{1}{\frac{10}{7}} = .7$$

~~11)~~ C = has cancer $P(C) = .005$

T = test says has cancer $P(T) = .95$

$P(C/T) = \frac{P(C \cap T)}{}$

12) A = IN ATHLETICS $P(A) = 1/100$

B = G.P.A. > 3 $P(B/A) = 1/10$

$P(\bar{A}) = 99/100$ $P(B/\bar{A}) = 2/10$

b) FIND $P(A/B)$

$$P(A/B) = \frac{P(A)\,P(B/A)}{P(A)\,P(B/A) + P(\bar{A})\,P(B/\bar{A})}$$

$$= \frac{\frac{1}{1000}}{\frac{1}{1000} + \frac{198}{1000}} = \frac{1}{199}$$

c)

Pg 53-4 10-6-70

1) $P(A \cap B) = P(A)P(B)$ $A \cap B = 0$

YES

2) $P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$ YES

3) $P(A \cap B) = 0$ NO!

4) YES

5) $P(A) = \frac{1}{6}$ $P(B) = \frac{1}{6}$ $P(C) = \frac{1}{6}$ NO

6) $P(A) = \frac{1}{2}$ $P(B) = \frac{4}{8} = \frac{1}{2}$ $P(C) = \frac{1}{2}$

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

$P(A \cap B \cap C) = \frac{1}{8} = P(A)P(B)P(C)$

ETC

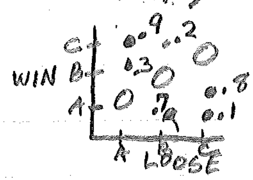
7) A & B IND. $\Rightarrow P(A \cap B) = P(A)P(B)$

TBP $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

8) P(MAKES SHOT) = .7

a) $(.7)^{15}$

b) $\binom{15}{4}(.7)^{14}(.3)$ OH YEAH!

9) 

A = EVENT A WINS THE TOURN.

$P(A_B B) = .7$ $P(B_B A) = .3$

$P(B_B C) = .8$ $P(C_B B) = .2$

$P(C_B A) = .9$ $P(A_B C) = .1$

a) $P(\text{A wins}) = P(A_B B) \cdot P(A_B C)$

$= (.7)(.1) = .07$

b) $P(\text{B wins}) = P(B_B C) \cdot P(B_B A)$

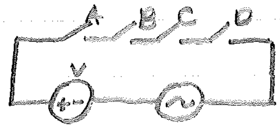
$= (.8)(.3) = .24$

c) $P(\text{C wins}) = (.9)(.2) = .18$

$P(\text{NOBODY WINS}) = 1 - .18 - .24 - .07$

$= 1 - .49 = .51$

AUX. PROB. 10-6-70

7) 

P(FAIL) = P(A FAILS OR B FAILS OR C FAILS OR D FAILS)

P(WORK) = P(A WORKS AND B WORKS AND C WORKS AND D WORKS)

A,B,C,D ARE IND. $\Rightarrow \overline{A}, \overline{B}, \overline{C}, \overline{D}$ ARE IND

P(WORK) = P(A WORKS) P(B WORKS) P(C WORKS) P(D WORKS)

$= (.98)^4 = .921$

P(FAIL) = 1 − P(WORKS) = .079

9) a) b) —Ⓐ—Ⓑ—

P(DEF) = P(A DEF OR B DEF)

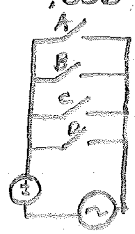
P(WORK) = P(A WORK AND B WORK)

= P(A WORK) P(B WORK)

$= (.975)(.989) = .964$

P(DEF) = 1 − P(WORK) = .036

c) $\frac{.011}{.036} = .306$

8) 

P(FAIL) = P(A FAIL) P(B FAIL) P(C FAIL) P(D FAIL)

$= (.02)^4$

$= 8 \times 10^{-8}$

16) $A_1 \{4G, 2B\}$ $A_2 \{4G, 2B)$

$\frac{\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15} = \frac{2}{5}$

$\left(\frac{2}{5}\right)^2 = \frac{4}{25}$

17) $S = \{(D,E), (D,T) (E,T), (P)\}$

a) $A = \{(D,E), (D,T)\}$ $P(A) = 1/2$

b) $B = \{(D,T), (E,T)\}$ $P(B) = 1/2$

c) $C = \{(D,E)(E,T)\}$ $P(C) = 1/2$

$P(A \cup B) = \frac{1}{4} = P(A)P(B) = 1/4$

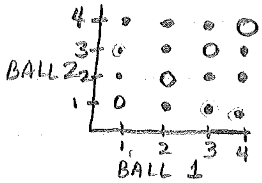
$P(A \cup C) = \frac{1}{4} = P(A)P(C) = 1/4$

$P(B \cup C) = \frac{1}{4} = P(B)P(C) = 1/4$

$P(B \cup A \cup C) = 0 \neq P(B)P(A)P(C) = 1/64$

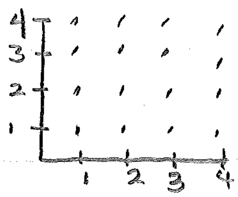
Pp. 67-8 10-12-70 3.1

1) $P(Y) = 1/4 \; ; \; Y = 1, 2, 3, 4$

2) 

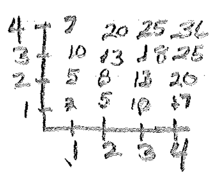
Z	P(Z)
3	2/12
4	2/12
5	4/12
6	2/12
7	2/12

$$P(Z)\begin{cases} = \frac{1}{6} & ; \; Z = 3, 4, 6, 7 \\ = \frac{1}{3} & ; \; Z = 5 \end{cases}$$

3) 

Z	P(Z)
2	1/16
3	2/16
4	3/16
5	4/16
6	3/16
7	2/16
8	1/16

$$P(Z) = \begin{cases} \frac{Z-1}{16} & ; \; Z = 1, 2, 3, 4, 5 \\ \frac{9-Z}{16} & ; \; Z = 5, 6, 7, 8 \end{cases}$$

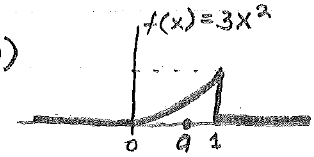
4) 

Z	P(Z)
2	1/16
5	2/16
7	2/16
8	1/16
10	2/16
13	2/16
18	1/16
20	2/16
25	2/16
36	1/16

$$P(Z)\begin{cases} \frac{1}{16} \; ; & Z = 2, 8, 18, 36 \\ \frac{1}{8} \; ; & Z = 5, 7, 10, 13, 20, 25 \end{cases}$$

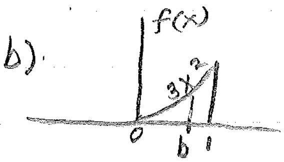
AUX. PROB.

28) a)



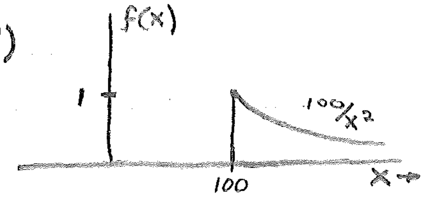
$$\int_0^a 3x^2dx = \int_a^1 3x^2dx$$
$$x^3\Big|_0^a = x^3\Big|_a^1$$
$$a^3 = 1 - a^3$$
$$2a^3 = 1 \Rightarrow a = \sqrt[3]{\tfrac{1}{2}}$$

b)



$$\int_b^1 3x^2dx = .05$$
$$x^3\Big|_b^1 = .05$$
$$1 - b^3 = .05 \Rightarrow b^3 = .95 \Rightarrow b = \sqrt[3]{.95}$$

29) a)



P(B) = P(ONE TUBE WILL BURN OUT IN 150 HRS)

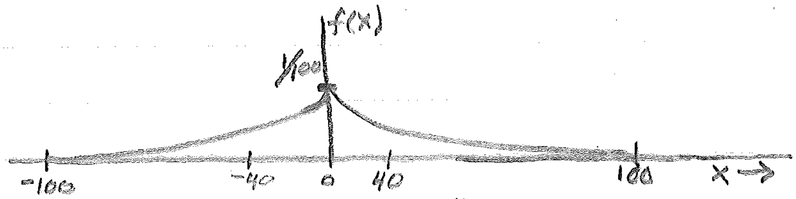
$$P(B) = \int_{100}^{150} \frac{100}{x^2}dx = -100x^{-1}\Big|_{100}^{150}$$
$$= -100\left[\frac{1}{150} - \frac{1}{100}\right]$$
$$= -\frac{2}{3} + 1 = \frac{1}{3} \Rightarrow P(\bar{B}) = \frac{2}{3}$$

P(C) = P(3 TUBES HAV'N'T BURNT OUT IN 150 HRS)

$$P(C) = \left(\tfrac{2}{3}\right)^3 = \tfrac{8}{27}$$

b) $P(3B) = \left(\tfrac{1}{3}\right)^3 = \tfrac{1}{27}$

31)



$$P(1 \text{ BOMB HITS}) = 2\int_0^{40} \frac{100+x}{10000}dx$$
$$= 2\left[\frac{x}{100} + \frac{x^2}{20000}\right]_0^{40}$$
$$= 2\left[\frac{40}{100} + \frac{1600}{20000}\right]$$
$$= 2\ \frac{96}{200} = \frac{96}{100} = \frac{48}{50} = \frac{24}{25}$$
$$\frac{1}{25^3}$$

P(H) = P(1 AT LEAST OF 3 HITS)

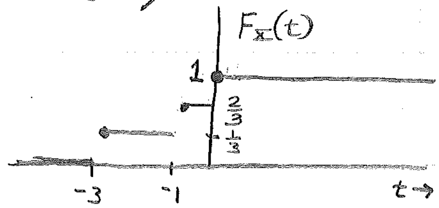
$P(\bar{H})$ = P(NO BOMB HITS) $= \frac{1}{25^3} = .641 \times 10^{-4}$

$\therefore P(H) = 1 - \frac{1}{25^3} = .9999359$

Pg 77-80 10-23-70

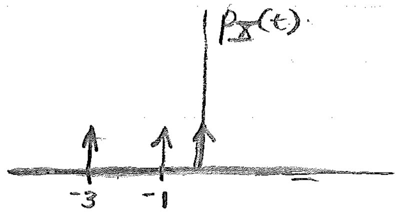
(3.2)

1)



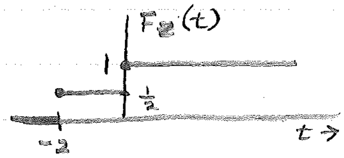
$\lim_{t\to\infty} F_X(t)=0$

$\lim_{t\to-\infty} F_X(t)=1$



$P_X(t)=\frac{1}{3}$; $t = -3, -1, 0$

2)

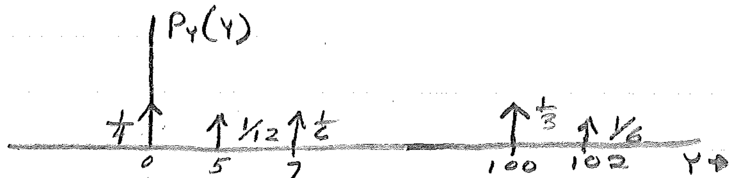


$\lim_{t\to\infty} F_Z(t)=1$; $\lim_{t\to-\infty} F_Z(t)=0$

$P_Z(Z)=\frac{1}{2}$; $Z=-2, 0$

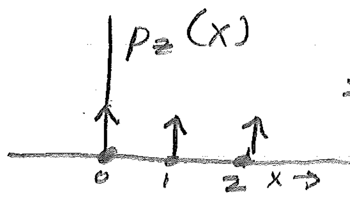
4) $\lim_{t\to\infty} F_W(t)=0$; $\lim_{t\to+\infty} F_W(t)=1$

W	0	5	7	100	102
$p_W(W)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

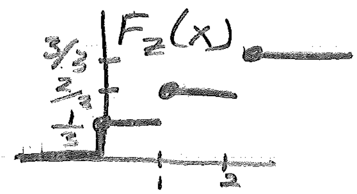


$P(Y\le 100)=1-P(Y=102)$
$= 5/6$

5)



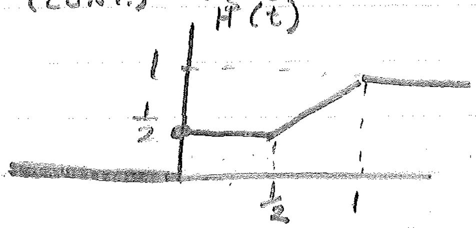
⇒



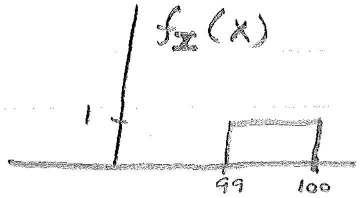
$$F_Z(x) = \begin{cases} 0 & x<0 \\ 1/3 & 0\le x<1 \\ 2/3 & 1\le x<2 \\ 1 & 2\le x \end{cases}$$

6)
$$F_U(x) = \begin{cases} 0 & t<-3 \\ \frac{1}{2} & -3\le t<0 \\ \frac{4}{6} & 0\le t<4 \\ 1 & t\ge 4 \end{cases}$$

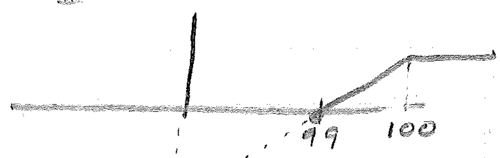
11) (CONT.) Pg 80



12)



$F_X(x) = \int 1 \cdot dx = x + c$



$F_X(x) = x + c$

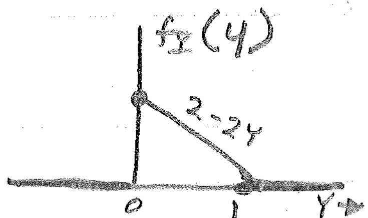
$0 = 99 + c \Rightarrow c = -99$

$F_X(x) = x - 99 \qquad 99 < x < 100$

$= 0 \qquad x < 99$

$= 1 \qquad x \geq 100$

13)



$\int 2 - 2Y = 2Y - Y^2 + C = F_Y(y)$

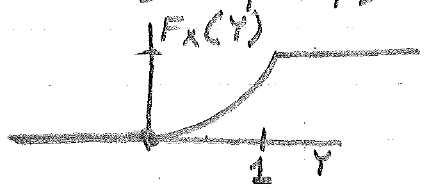
$Y = 1 \Rightarrow F_Y = 1$

$1 + C = 1 \Rightarrow C = 0$

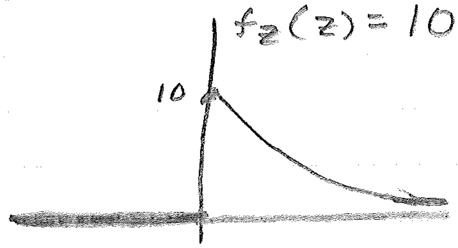
$\therefore F_Y(y) = 2Y - Y^2 \qquad 0 \leq Y \leq 1$

$= 0 \qquad Y \leq 0$

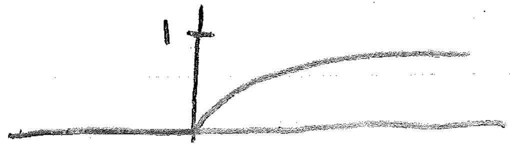
$= 1 \qquad Y \geq 1$



14) $f_Z(z) = 10e^{-10z}\mu(t)$



$\int 10e^{-10z} dz \Rightarrow -e^{-10z} + 1$

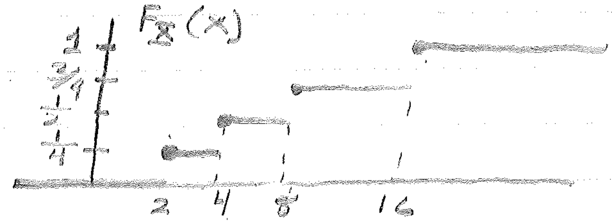


$F_Z(z) = 1 - e^{-10z} \qquad z \geq 0$

$= 0 \qquad z \leq 0$

P_{P1} 88-70 10-23-70 3-3

1) $p_{\underline{X}}(x) = \frac{1}{4} \quad x = 2, 4, 8, 16$
$\quad = 0 \quad$ OTHERWISE



a) $E(\underline{X}) = \sum H(x)\, p_{\underline{X}}(x)$
$= \frac{1}{4}(2+4+8+16) = \frac{30}{4} = \frac{15}{2}$

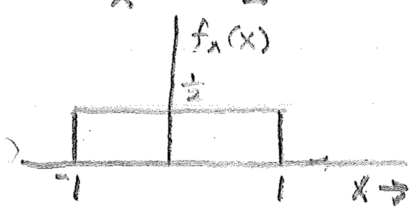
b) $E(\underline{X}^2) = \sum H(x^2)\, p_{\underline{X}}(x)$
$= \frac{1}{4}(4+16+64+256) = 1+4+16+64 = 85$

c) $E\left(\frac{1}{\underline{X}}\right) = \sum H\left(\frac{1}{x}\right) p_x(x)$
$= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right)$
$= \frac{1}{4}\left(\frac{8}{16} + \frac{4}{16} + \frac{2}{16} + \frac{1}{16}\right) = \frac{15}{64}$

d) $E\left(2^{\underline{X}/2}\right) = \frac{1}{4}\left(2^1 + 2^2 + 2^4 + 2^8\right)$
$= \frac{1}{4}(2+4+16+256) = \frac{278}{4} = \frac{139}{2}$

e) $\sigma_{\underline{X}}^2 = E(\underline{X}^2) - \mu_x^2$
$= 85 - \frac{225}{4} = \frac{340}{4} - \frac{225}{4} = \frac{115}{4}$
$\sigma_x = \frac{\sqrt{115}}{2}$

2)



a) $E(x) = \int_{-1}^{1} \left(\frac{1}{2}\right) x\, dx$
$\left[\frac{1}{4}x^2\right]_{-1}^{1} = \frac{1}{4}(1-1) = 0$

b) $f_{\underline{X}}(x) = \frac{1}{2} \quad -1 < x < 1$
$\quad = 0 \quad$ OTHERWISE
$E_{\underline{X}}(X) = \int_{-1}^{1} \frac{1}{2} x^2\, dx$
$= \frac{1}{2}\,\frac{1}{3}\, x^3$
$= \frac{1}{6}\left[1+1\right] = \frac{1}{3}$

c) $E_{\underline{X}}(x+2) = \int_{-1}^{1} \frac{1}{2}(x+2)\, dx = \int_{-1}^{1}\left(\frac{x}{2} + 1\right) dx$
$= \left(\frac{1}{4}x^2 + x\right]_{-1}^{1} = \frac{5}{4} + \frac{3}{4} = 2$

d) $E\left(\frac{\underline{X}}{4} + 7\right) = \int_{-1}^{1}\left(\frac{x}{8} + \frac{7}{2}\right) dx = \frac{x^2}{16} + \frac{7x}{2}\Big]_{-1}^{1} = \frac{1}{16} + \frac{7}{2} - \frac{1}{16} + \frac{7}{2} = 7$

e) $\sigma_x^2 = E(\underline{X}^2) - \mu_x^2 = \frac{1}{3} - 0 = \frac{1}{3}$

f) $= \sigma_x = \sqrt{1/3}$

Pg 104-105   10-24-70   3.5

3) $F_Y(t) = F_X\left(\frac{t-a}{b}\right)$

$\quad = F_X\left(\frac{t-15}{2}\right)$

$$F_Y(t) \left\{ \begin{array}{ll} = 0 & \frac{t-15}{2} < -1 \\ = \left(\frac{t-15}{2}+1\right)\frac{1}{2} & -1 \leq \frac{t-15}{2} \leq 1 \\ = 1 & \frac{t-15}{2} > 1 \end{array} \right.$$

$$\left\{ \begin{array}{ll} = 0 & t-15 < -2 \Rightarrow t < 13 \\ = \frac{t-15}{4} + \frac{1}{2} & -2 \leq t-15 \leq 2 \Rightarrow 13 \leq t \leq 17 \\ = 1 & t > 17 \end{array} \right.$$

$$\begin{array}{lll} F_Y(t) & = 0 & t < 13 \\ & = \frac{t-13}{4} & 13 \leq t \leq 17 \\ & = 1 & t > 17 \end{array}$$

4) $F_Z(t) = F_W\left(\frac{t-a}{b}\right)$

$\quad = F_W(t+1)$

$$F_Z(t) \left\{ \begin{array}{ll} = 0 & t+1 < 0 \\ = (t+1)^3 & 0 \leq t+1 \leq 1 \\ = 1 & t+1 > 1 \end{array} \right.$$

$$\begin{array}{lll} F_Z(t) & = 0 & t < -1 \\ & = (t+1)^3 & -1 \leq t \leq 0 \\ & = 1 & t > 0 \end{array}$$

$$\begin{array}{lll} f_Z(t) & = 0 & t < -1 \\ & = 3(t+1)^2 & -1 \leq t \leq 0 \\ & = 0 & t > 0 \end{array}$$

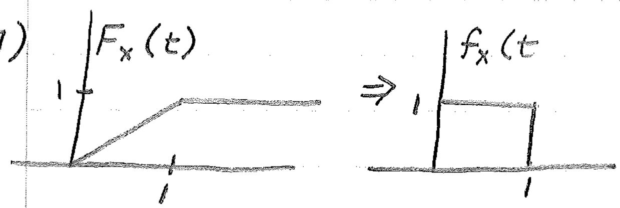
5) $F_U(t) = F_X\left(\frac{t+50}{7}\right)$

$$\begin{array}{lll} F_U(t) & = 0 & \frac{t+50}{7} < -10 \Rightarrow t+50 < -70 \Rightarrow t < -120 \\ & = \frac{1}{4} & -10 \leq \frac{t+50}{7} < 0 \Rightarrow -70 \leq t+50 < 0 \Rightarrow -120 \leq t < -50 \\ & = \frac{3}{4} & 0 \leq \frac{t+50}{7} < 10 \Rightarrow 0 \leq t+50 < 70 \Rightarrow -50 \leq t < 20 \\ & = 1 & \frac{t+50}{7} \geq 10 \Rightarrow t+50 \geq 70 \Rightarrow t \geq 20 \end{array}$$

$$\begin{array}{lll} f_U(t) & = \frac{1}{4} & t = -120 \\ & = \frac{1}{2} & t = -50 \\ & = \frac{1}{4} & t = 20 \end{array}$$

Pg 105 10-26-70 3.5

7)



$\mu_x = \int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$

$E(x^2) = \int_0^1 x^2 dx = \frac{x^3}{3}\Big]_0^1 = \frac{1}{3}$

$\sigma^2 = E(x^2) - \mu_x^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \Rightarrow \sigma_x = \frac{1}{2\sqrt{3}}$

$Z = \left(x - \frac{1}{2}\right) 2\sqrt{3} = 2\sqrt{3}\,x - \sqrt{3}$

$\cancel{F_Z(t) = 0} \qquad \cancel{2\sqrt{3}t - \sqrt{3} < 0 \Rightarrow 2\sqrt{3}t < \sqrt{3} \Rightarrow t < \frac{1}{2}}$

$F_Z(t) = F_x\left(\frac{t+\sqrt{3}}{2\sqrt{3}}\right)$

$F_Z(t) = 0 \qquad \frac{t+\sqrt{3}}{2\sqrt{3}} < 0 \Rightarrow t < -\sqrt{3}$

$= \frac{t+\sqrt{3}}{2\sqrt{3}} \qquad 0 \le \frac{t+\sqrt{3}}{2\sqrt{3}} \le 1 \Rightarrow 0 \le t+\sqrt{3} \le 2\sqrt{3}$

$\Rightarrow -\sqrt{3} \le t \le 2\sqrt{3} - \sqrt{3}$

$\Rightarrow -\sqrt{3} \le t \le \sqrt{3}$

$= 1 \qquad t > \sqrt{3}$

$f_Z(t) = \frac{1}{2\sqrt{3}} \qquad -\sqrt{3} \le t \le \sqrt{3}$

8) $E(Z) = \int_0^\infty t e^{-t} dt$

$u = t \quad dv = e^{-t}dt$

$du = dt \quad v = -e^{-t}$

$E(Z) = -e^{-t}t\Big]_0^\infty + \int_0^\infty e^{-t} dt$

$= -e^{-t}\Big]_0^\infty = 1 = \mu_x$

$E[Z^2] = \int_0^\infty t^2 e^{-t} dt$

$du = 2t\,dt \quad v = -e^{-t}$

$E(Z^2) = -t^2 e^{-t}\Big]_0^\infty + 2\int_0^\infty t e^{-t}$

$= 2$

$\sigma_Z^2 = E(Z^2) - \mu_Z^2 = 2 - 1 = 1 \Rightarrow \sigma_Z = 1$

$\therefore \quad A = (Z - 1)/1 \Rightarrow F_A(t) = F_Z\left(\frac{t+1}{1}\right)$

$F_A(t) = 1 - e^{-(t+1)} \qquad t+1 > 0 \Rightarrow t > -1$

$= 0 \qquad t \le -1$

$f_A(t) = e^{-(t+1)} \qquad t > -1$

AUX. PROBLEMS 10-26-70

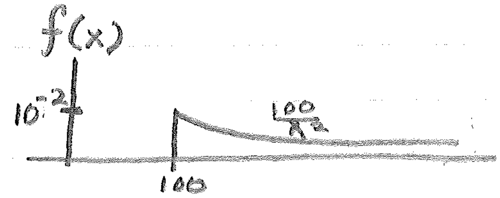
21)

c	$p(c)$	s	$p(s)$
1	1/3	50,000	1/10
2	2/3	0	9/10

$\mu_c = E(C) = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}\ \frac{\text{CUS}}{\text{DAY}}$

$\mu_S = E(S) = \frac{50,000}{10} = 5000\ \frac{\text{DOL}}{\text{CUST}}$

$(\$5000/\text{CUST})\ \frac{5}{3}\ \frac{\text{CUST}}{\text{DAY}} = \frac{\$25000}{3\ \text{DAY}} = \$8333/\text{DAY}$

29) 

$P(1\text{ REPLACED}) = \int_{100}^{150} \frac{100}{x^2}dx = \frac{-100}{x}\Big]_{100}^{150} = \frac{-100}{150} + \frac{100}{100} = \frac{1}{3}$

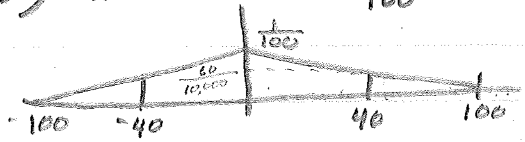
a) $P(\text{NOT REPLACED}) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(1\text{ N.R. AND } 2\text{ N.R. AND } 3\text{ N.R.}) = P(1\text{ N.R})P(2\text{ N.R})\ P(3\text{ NR})$

(ASSUMING INDEPENDENCE) $= \left(\frac{2}{3}\right)^3 = 8/27$

b) $\left(\frac{1}{3}\right)^3 = 1/27$

c) $\mu_x = E(X) = \int_{100}^{\infty} x\ \frac{100}{x^2}dx = 100\int_{100}^{\infty}\frac{1}{x}dx$

31) 

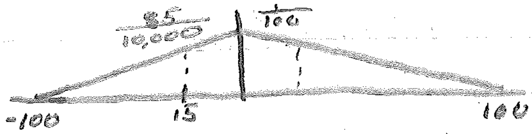
$P(\text{DAMAGE}) = 1 - P(\text{NONE})$

$P(\text{MISS}) = 2\int_{40}^{100}\left(\frac{1}{100} - \frac{x}{10,000}\right)dx = 2\left[\frac{x}{100} - \frac{x^2}{20,000}\right]_{40}^{100}$

$= \frac{60}{10,000}\ 60 = .36$

$P(\text{NONE}) = [P(\text{MISS})]^3$

$\therefore P(\text{DAMAGE}) = 1 - (.36)^3$

32) 

$P(\text{MISS}) = \frac{85}{10,000}\ 85$

$= (.7225)$

$\therefore P(\text{DAMAGE}) = 1 - (.7225)^{10}$

Pg 112 11-2-70 4.1

2) $p = \frac{8}{10}$; $q = \frac{2}{10}$; $n = 20$

a) $\mu_Y = pn = 16$

b) $\sigma_Y^2 = (.8)(.2)(20) = 3.2 \Rightarrow \sigma_Y = 1.79 = \sqrt{\frac{16}{5}}$

c) $P(Y=16) = P(Y \leq 16) - P(Y \leq 15)$

LET $P(Z) = P($Z BLACK BALLS ARE PICKED$)$

$n = 20,\ p = \frac{2}{10},\ q = \frac{8}{10}$

$P(Z=4) = P(Z \leq 4) - P(Z \leq 3)$

$= .6296 - .4114 = .2182$

d) $P(Y<14) = \cancel{P(Y \leq 13) = P(Z \geq 14) = P(Z \leq 15) - P(Z \leq 13)}$

$\cancel{= P(Z > 13) = P(Z \geq 14) = P(Z \leq 20) - P(Z \leq 13)}$

$= P(Z \geq 6) = P(Z \leq 20) - P(Z \leq 6)$

$= 1.0000 - .9133$

$= .0867$

e) $P(Y > 18) = P(Z \leq 2) = .2061$

3) Z = # OF COINS LANDING HEADS UP

$n = 10$; $p = q = \frac{1}{2}$

a) $P(Z=5) = P(Z \leq 5) - P(Z \leq 4)$

$= .6230 - .3770 = .2460$

b) $\mu_Z = np = 10 \frac{1}{5} = 5$

4) X = # OF DEF. GLASSES

$p = .01 \quad n = 10$

a) $P(X=0) = \binom{10}{0}(.01)^0(.99)^{10} = .9044$

b) $E(X) = \mu_X = np = (10)(.01) = .1$

5)

Pg 119-120 11-10-70

1) $p_x(k) = (.5)^{k-1}(.5) = .5^k$

$p_x(X \le 2) = .5^2 + .5^1 = .25 + .5 = .75$

$p_x(X \le 3) = .75 + .5^3 = .750 + .125 = .875$

2) 1) $p = \frac{18}{38}$

$P_x(k) = \left(\frac{20}{38}\right)^{k-1}\left(\frac{18}{38}\right)$

2) $\mu_x = \frac{1}{p} = \frac{38}{18}$

3) $p = \frac{2}{38}$

a) $P_Y(k) = \left(\frac{36}{38}\right)^{k-1} \frac{2}{38}$

b) $\mu_Y = \frac{1}{p} = 19$

c) $\sigma_Y^2 = \frac{q}{p^2} = \left(\frac{36}{38}\right)\frac{38^2}{4} = 342$

4) $p = \frac{1}{10}$

a) $P_Z(k) = \left(\frac{9}{10}\right)^{k-1}\frac{1}{10}$

b) $\mu_x = \frac{1}{p} = 10$

5) X $W = 1$ $M = 10$

$P_Y(k) = \frac{\binom{1}{k}\binom{9}{n-k}}{\binom{10}{n}}$ $k = 0, 1 \Rightarrow n = 1, 2, 3 \ldots 10 \Leftrightarrow P_Y(k) \ge 0$

6) $M = 5$ $W = 3$ $n = 2$

a) $P_M(k) = \frac{\binom{3}{k}\binom{2}{2-k}}{\binom{5}{2}}$

7) $M = 52$ $W = 26$ $n = 13$

a) $P_c(k) = \frac{\binom{26}{k}\binom{26}{13-k}}{\binom{52}{13}}$

b) $\mu_x = \frac{nW}{M} = \frac{13 \cdot 26}{52} = \frac{13}{2}$

c) $\sigma_c^2 = n\left(\frac{W}{M}\right)\left(\frac{M-W}{M}\right) \cdot \frac{(M-n)}{M-1}$

$= 13 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{39}{51} = \frac{169}{68}$

Pg 128-129 4.3 11-10-70

1) $\lambda = \frac{3}{HR}$ $s = \frac{1}{2} HR \Rightarrow \lambda s = \frac{3}{2}$

$p_x(k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s} = \frac{(\frac{3}{2})^k}{k!} e^{-1.5}$

a) $P_x(0) = e^{-1.5} = .223$

b) $P(x \geq 2) = 1 - P(x=1) - P(x=0)$

$P(x=1) = \frac{\frac{3}{2}}{1!} e^{-1.5} = .335$

$\Rightarrow P(x \geq 2) = 1 - .335 - .223 = .442$

2) a) $\lambda = \frac{8}{HR}$ $s = 1 \Rightarrow \lambda s = 8$

$P_x(k) = \frac{8^k}{k!} e^{-8}$

$P_x(k=0) = e^{-8} = .00034$

b) $\lambda = \frac{8}{HR}$ $s = \frac{1}{4} HR \Rightarrow \lambda s = 2$

$P_x(k) = \frac{2^k}{k!} e^{-2}$

$P_x(0) = e^{-2} = .135$

c) $(.135)^4 = .0003$

3) $\lambda = \frac{10}{HR}$ $s = \frac{1}{6\,HR} \Rightarrow \lambda s = \frac{10}{6} = \frac{5}{3}$

$P_x(k) = \frac{(\frac{5}{3})^k}{k!} e^{-\frac{5}{3}}$

$P_x(1) = \frac{5}{3} e^{-\frac{5}{3}} = .811$

b) $(.811)^3$

4) $\lambda = \frac{1}{2\,MONTHS}$

a) $\mu_x = 6$

b) $\sigma_x^2 = 6 \Rightarrow \sigma_x = \sqrt{6}$

c) $P_x(k) = \frac{(\frac{1}{2})^k}{k!} e^{-\frac{1}{2}}$

$P_x(0) = e^{-\frac{1}{2}} = .607$

5) $P_x(\lambda s) = \frac{(\lambda s)^k}{k!} e^{-\lambda s}$

$P_s(\lambda s - 1) = \frac{(\lambda s - 1)^k}{k!} e^{-\lambda s + 1}$

8) $\lambda = \frac{1 \text{ BLIND}}{10^4 \text{ BABIES}}$; $s = 5000$ BABIES $\Rightarrow \lambda s = 5 \times 10^{-1}$

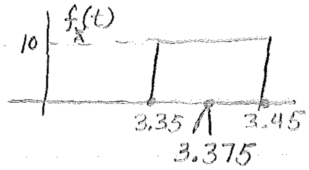
$P(\lambda s) = \frac{(.5)^k}{k!} e^{-.5}$

$P(1) = \frac{1}{2} e^{-.5}$

$P(2) = \frac{1}{2} \frac{1}{4} e^{-.5}$

Pg 136-137 4.4 11-12-70

5) $\lambda = \frac{1}{4}\ lb \quad \frac{1}{s} = 3\frac{3}{8} = \frac{27}{8} \Rightarrow \lambda s = \frac{2}{27}\ \frac{lb}{inch}$



$\frac{1}{4}\ lb \cdot \frac{27}{2}\ \frac{inch}{lb} = \frac{27}{8}\ inch$

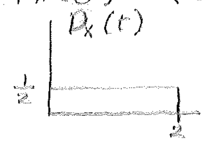
$p(X \geq 3.375'') = p(W \geq \frac{1}{4}) = 10[3.450 - 3.375]$

$= 10(.075) = .75$

a) $(.75)^4 = .316$

b) $P_X(3) = \binom{4}{3}(.75)^3(.25) = .422$

6)

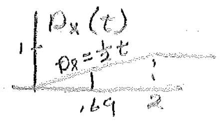


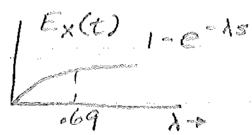
$\frac{1}{2}\lambda = 1 - e^{-\lambda s}$

$e^{-\lambda s} = \frac{2-\lambda}{2}$

$-\lambda s = \ln \frac{2-\lambda}{2}$

ARG!





7) $F_X(t) = 1 - e^{-5t} = P(X < t)$

a) $P(X \geq \frac{1}{4}) = 1 - P(X \leq \frac{1}{4})$

$P(X \leq \frac{1}{4}) = 1 - e^{-\frac{5}{4}} = 1 - e^{-1.25} = 1 - .286$

$\therefore P(X \geq \frac{1}{4}) = 1 - 1 + .286 = .286$

b) $P(X \leq \frac{1}{6}) = 1 - e^{-\frac{5}{6}} = 1 - .434 = .566$

c) 0

8) $\lambda = \frac{50}{HR}$

a) $F_X(t) = P(X \leq t) = 1 - e^{-50t}$

$P(X \geq 1/30) = 1 - P(X \leq \frac{1}{30}) = e^{-\frac{5}{3}} = .1882$

b) SAME AS ABOVE: .1882

9) $\sigma_X^2 = \frac{16}{12} = \frac{4}{3}$

$\sigma_Y^2 = \sigma_X^2 = \frac{4}{3} = \frac{1}{\lambda^2} \Rightarrow \lambda^2 = \frac{3}{4} \Rightarrow \lambda = \sqrt{3/4}$

11-13-70 Pg 143-4 4.5

1) a) $F_x(t) = \int_{-\infty}^{\frac{t-241}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-(t-241)^2}{8}} dz = N_z\left(\frac{t-241}{2}\right)$

$P(X < 240) = \int_{-\infty}^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{8}} dz = .3085$

b) $P(X > 235) = 1 - P(X < 235)$

$N_z\left(\frac{-6}{2}\right) = .0013 \Rightarrow 1 - .0013 = .9987$

2) $\mu = 16$; $\sigma = 1$

$N_z\left(\frac{t-16}{1}\right) = N_z(t-16)$

a) $P(X > 14) = 1 - P(X < 14) = 1 - N_z(-2) = 1 - .0227 = .9773$

b) $P(X < 17) = N_z(1) = .8413$

c) $P(12 < X < 15) = P(X < 15) - P(X < 12)$

$= N_z(-1) - N_z(-4) = .1587 - 0 = .1587$

3) $F_U(t) = 2N_z(t) - 1$

AUX. PROBS.

22) $p = 1/10$ $\qquad p_x(x) = \binom{n}{x} p^x q^{n-x}$

$$p_x(2) = \binom{30}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{28}$$

23) $\lambda = \frac{3 \text{ RAINS}}{30 \text{ DAYS}}$ $\qquad S = 2 \text{ DAYS} \Rightarrow \lambda s = \frac{1}{5}$

$$p_x(2) = \frac{(.2)^2}{2!} e^{-.2}$$

24) $\lambda s = \frac{500}{365}$

$$p_x(1) = \left(\frac{500}{365}\right) e^{-\frac{500}{365}}$$

38) $\mu = 60$; $\sigma = 13 \Rightarrow N_z\left(\frac{t-60}{13}\right)$

a) $P(z > 100) = 1 - p(z < 100) = 1 - N_z\left(\frac{40}{13}\right) = 1 - N_z(3.1) = 0$

b) $P(z > t) = .05 \Rightarrow N_z\left(\frac{t-60}{13}\right) = .95 = N_z(1.65)$

$\therefore \frac{t-60}{13} = 1.65 \Rightarrow t-60 = 21.65 \Rightarrow t = 81.65$

c) $P(z < 45) = N_z\left(\frac{-15}{13}\right) = N_z(-1.2) = .115 \Rightarrow 115$ FAIL

64) $\lambda = \lambda \frac{PAR}{HR}$

$$f_x(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0, 1, 2, \ldots, 29$$

$$= 1 - \sum_{x=0}^{29} \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 30$$

Pg 150   11-19-70   5.1

1)

X	1	2
1	2	3
2	3	4

Y	1	2
1	0	-1
2	1	0

$P_{XY}(X,Y)$	2	3	4
-1	0	1/4	0
0	1/4	0	1/4
1	0	1/4	0

2)

111	211
112	212
121	221
122	222

	$P_{XY}(X,Y)$	Y: 2	3	4
	2	1/8	1/8	0
X	3	1/8	1/4	1/8
	4	0	1/8	1/8

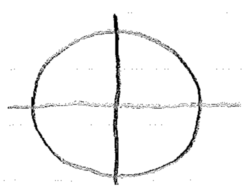
4) $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$

		X: 0	1	2
Y	0	0	0	$\frac{3}{10}\cdot\frac{2}{9} = \frac{6}{90}$
	1	0	$\frac{\binom{3}{1}\binom{7}{1}}{\binom{10}{2}} = \frac{42}{90}$	0
	2	$\frac{7}{10}\cdot\frac{6}{9} = \frac{42}{90}$	0	0

$P(X>Y) = 1 - P(Y \geq X) = \frac{6}{90}$

6) $\int_1^2 \int_0^1 A \frac{X}{Y}\, dx\, dY = 1$

$\int_1^2 \frac{A}{2}\frac{X^2}{Y}\Big]_0^1 dY = \int_1^2 \frac{A}{2}\frac{1}{Y}\, dY = \frac{A}{2} \ln Y\Big]_1^2 = \frac{A \ln 2}{2} = 1 \Rightarrow A = \frac{2}{\ln 2}$

~~5)~~ 

$Y^2 + X^2 = 1$ $f_{XY}(X,Y) = C$ $-1 < X < 1$ $-\sqrt{X^2+1} < Y < \sqrt{X^2+1}$

$= 0$ OTHERWISE.

$C \int_{-1}^{1} \int_{-\sqrt{1-X^2}}^{\sqrt{1-X^2}} dY\, dX$

$C \int_{-1}^{1} 2\sqrt{1-X^2}\, dX$

$-4X$

Pg 158-9

1)
-1 -1
-1 1
1 -1
+1 1

		X	
P(XY)		-1	1
Y	-1	1/4	1/4
	1	1/4	1/4

$\Rightarrow P(X,Y) = \frac{1}{4} \quad X = -1, 1 \,;\, Y = -1, 1$

$P_X = 1/2 \quad X = -1, 1$

$P_Y = 1/2 \quad Y = -1, 1$

4) $P_{W,Z}(w,z) = 1/n$ FOR $(w,z) = (1,1), (2,2), \ldots, (n,n)$

$= 0$ OTHERWISE

a) $E(W/Z) = \sum_{\text{ALL } W} \sum_{\text{ALL } Z} \frac{w}{z} p_{W(Z)}(w,z)$

$= \sum_{1}^{n} \frac{w}{z} \frac{1}{n} = 1$

b) $E(Z/W) = 1$

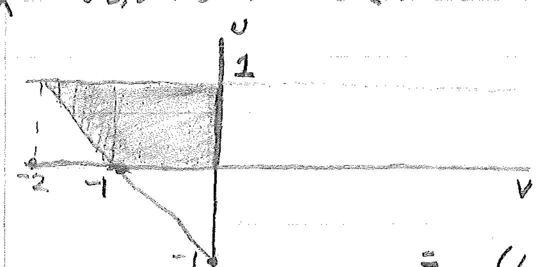
c) $E[W^2 + Z^2] = \sum_{i=0}^{n} \frac{1}{n}[i^2 + i^2] = \sum_{i=0}^{n} \frac{2}{n} i^2 = \frac{2}{n} \sum_{i=0}^{n} i^2$

$= \frac{2}{n} \frac{1}{6} n(n+1)(2n+1)$

$= \frac{1}{3}(n+1)(2n+1)$

d) $E[W^3/Z^2] = \sum_{i=0}^{n} i \frac{1}{n} = \frac{1}{n} \sum_{i=0}^{n} i = \frac{1}{n} \frac{1}{2} n(n+1) = \frac{1}{2}(n+1)$

~~5)~~ $f_{U,V}(u,v) = 6(1-u-v) \quad 0 < u < 1, \; 0 < v < 1-u$



$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv$

$= \int_{1-u}^{1} (6 - 6U - 6V)\,dV \quad -2 < V < -1$

$+ \int_{0}^{1} (6 - 6U - 6V)\,dV \quad -1 < V < 0$

$= (6-6U)V - 3V^2 \Big]_{1-U}^{1} + (6-6U)V - 3V^2 \Big]_{0}^{1}$

$= (6-6U) - 3 - (6-6U)(1-U) + 3(1-U)^2 + 6 - 6U - 3$

$= \cancel{6} - 6U - \cancel{3} + (6-6U)U + \cancel{3} - 6U + 3U^2 - \cancel{3}$

$= 3 - 6U - 3U^2 = 3$

Pg 159 5.2 11-20-90

7)

		X		
	-3	-1	1	3
-5	1/4	0	0	0
-1	0	1/4	0	0
1	0	0	1/4	0
5	0	0	0	1/4

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\mu_X = \sum_{\text{ALL } X} \sum_{\text{ALL } Y} X\, p_{XY}(x,y)$$
$$= (-3)(1/4) - (1)\tfrac{1}{4} + (1)\tfrac{1}{4} + 3(1/4) = 0$$

$$E(X^2) = \sum_{\text{ALL } X} \sum_{\text{ALL } Y} X^2 p_{XY}(x,y)$$
$$= (9)\,1/4 + (1)\,1/4 + (1)\,1/4 + 9\,\,1/4 = \frac{20}{4}$$

$$\Rightarrow \sigma_X^2 = 5 \Rightarrow \sqrt{\sigma_X^2} = \sigma_X = \sqrt{5}$$

$$\mu_Y = \sum \sum Y\, p_{XY}(x,y)$$
$$= (-5)\tfrac{1}{4} + (-1)\,1/4 + 1(1/4) + 5(\tfrac{1}{4}) = 0$$

$$E(Y^2) = \sum\sum Y^2 p_{XY}(x\,y)$$
$$= (25)\tfrac{1}{4} + (1)\tfrac{1}{4} + (1)\tfrac{1}{4} + 25(\tfrac{1}{4}) = \frac{52}{4} = 13$$

$$\Rightarrow \sigma_Y = \sqrt{13}$$

$$\sigma_{XY} = E(XY) - \cancel{\mu_X \mu_Y}$$
$$E(XY) = 15(\tfrac{1}{4}) + 1(\tfrac{1}{4}) + 1(\tfrac{1}{4}) + 15(\tfrac{1}{4}) = \frac{32}{4}$$
$$\Rightarrow \sigma_{XY} = 32/4$$

$$\therefore \rho_{XY} = \frac{8}{\sqrt{65}}$$

Pg 169 5.3 11-20-70

1) $p_{X,Y}(X,Y) = \frac{1}{n^2}$ $X = 1, 2, 3, \dots, n$ $Y = 1, 2, 3, \dots, n$

$$E(X) = \sum_{X=1}^{n} \sum_{Y=1}^{n} \frac{X}{n^2}$$
$$= \sum_{X=1}^{n} \frac{X}{n} = \frac{1}{n} \sum_{X=1}^{n} X = \frac{1}{\cancel{n}} \frac{1}{2} \cancel{n}(n+1) = \frac{1}{2}(n+1)$$
$$E(Y) = \sum_{Y=1}^{n} \sum_{X=1}^{n} \frac{Y}{n^2} = \frac{1}{n} \sum_{Y=1}^{n} Y = \frac{1}{n} \frac{1}{2} n(n+1) = \frac{1}{2}(n+1)$$
$$E(XY) = \sum_{X=1}^{n} \sum_{Y=1}^{n} XY \frac{1}{n^2}$$
$$= \sum_{X=1}^{n} \frac{1}{n^2} X \frac{1}{2} \cancel{n}(n+1) = \frac{1}{2\cancel{n}}(n+1) \frac{1}{2} \cancel{n}(n+1)$$
$$= \frac{1}{4}(n+1)^2 = E(X)E(Y)$$

3) $f_{XY}(X,Y) = 4$ $0 < X < 1$, $0 < Y < 1/4$

$$E(X) = \int_0^1 \int_0^{1/4} 4X \, dY \, dX$$
$$= \int_0^1 4XY \Big]_0^{1/4} dX = \int_0^1 X \, dX = \frac{X^2}{2}\Big]_0^1 = \frac{1}{2}$$
$$E(Y) = \int_0^{1/4} \int_0^1 4Y \, dX \, dY$$
$$= \int_0^{1/4} 4XY \Big]_0^1 dY = \int_0^{1/4} 4Y \, dY = 2Y^2 \Big]_0^{1/4} = \frac{1}{8}$$
$$E(XY) = 4\int_0^1 X \int_0^{1/4} Y \, dY \, dX$$
$$= 4\int_0^1 X \left[\frac{Y^2}{2}\right]_0^{1/4} dX = 4\int_0^1 X \frac{1}{32} \, dX$$
$$= \frac{1}{8}\int_0^1 X \, dX = \frac{1}{16} X^2 \Big]_0^1 = \frac{1}{16} = E(X)E(Y)$$

PROBLEM SESSION

Pg 169, #1

$$P_X(x) = \sum_{Y=1}^{n} \frac{1}{n^2} = \frac{n}{n^2} = \frac{1}{n} \quad \forall \quad x = 1, 2, 3, 4, \ldots, n$$

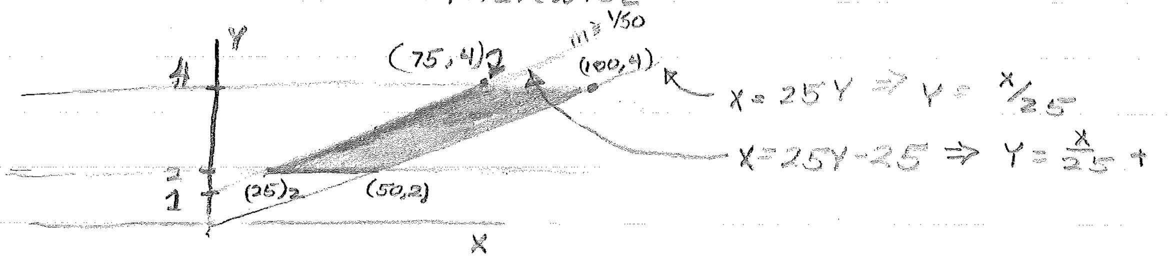
$$P_Y(Y) = \sum_{X=1}^{n} \frac{1}{n^2} \qquad = \frac{1}{n} \quad \forall \quad Y = 1, 2, 3, 4, \ldots, n$$

$$P_X(x) \cdot P_Y(Y) = \frac{1}{n^2} = p(X, Y) \Rightarrow X \,\&\, Y \text{ ARE INDEPENDENT}$$

Pg 1589 #2

$$f(x, Y) = \frac{1}{50} \quad \text{FOR} \quad 2 < Y < 4 \qquad 25(Y-1) < X < 25Y$$

$$= 0 \qquad \text{OTHERWISE}$$



$$f_Y(Y) = \int_{-\infty}^{\infty} f(x, Y)\, dx$$

$$= \int_{25(Y-1)}^{25Y} \frac{1}{50}\, dx = \frac{1}{2}Y - \frac{1}{2}(Y-1) = \frac{1}{2} \quad \text{FOR} \quad 2 < Y < 4$$

$$\Rightarrow f_Y(Y) = \begin{cases} \frac{1}{2} & 2 < Y < 4 \\ 0 & \text{OTHERWISE} \end{cases}$$

LET $25 < X < 50$ $\qquad f_X(x) = \int_2^{\frac{x}{25}+1} \frac{1}{50}\, dY = \frac{1}{50}\left[\frac{x}{25} - 1\right]$

$50 < X < 75$ $\qquad f_X(x) = \int_{\frac{x}{25}}^{\frac{x}{25}+1} \frac{1}{50}\, dY = \frac{1}{50}[1] = \frac{1}{50}$

$75 < X < 100$ $\qquad f_X(x) = \int_{x/25}^{4} \frac{1}{50}\, dY = \frac{1}{50}\left[4 - \frac{x}{25}\right]$

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{50}\left[\frac{x}{25} - 1\right] & ; \quad 25 < X < 50 \\ \frac{1}{50} & ; \quad 50 < X < 75 \\ \frac{1}{50}\left[4 - \frac{x}{25}\right] & ; \quad 75 < X < 100 \\ 0 & ; \quad \text{OTHERWISE} \end{cases}$$

$$\mu_Y = \int_{-\infty}^{\infty} Y f_Y(Y)\, dY = \int_2^4 \frac{Y}{2}\, dY = 3$$

$$\mu_X = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{50}\left[\int_{25}^{50} x\left(\frac{1}{25}x - 1\right) dx + \int_{50}^{75} x\, dx + \int_{75}^{100} x\left(4 - \frac{x}{25}\right) dx\right]$$

AFTER MUCH PLUGGING AND CHUGGING YIELDS

$$\mu_X = \frac{125}{2}$$

① Definitions, Axioms, & Theorems (pp 1-13)

1.1.1. DEF: Two sets A & B are equal iff every element in A is in B and every element of B is in A

1.1.2. DEF: A is a subset of B ($A \subset B$) iff every element in B is in A

1.2.1 DEF: The union of A & B ($A \cup B$) is the set containing all of the elements in A and in B or in both: $(A \cup B) = \{X \mid X \in A \text{ OR } X \in B\}$

1.2.2 DEF: The intersection of A & B ($A \cap B$) is the set containing all the elements common to both A & B: $A \cap B = \{X \mid X \in A \text{ AND } X \in B\}$

1.3.1 DEF: The complement of A, with respect to a universal set, is the set of elements not in A: $\overline{A} = \{X \mid X \notin A)$

1.3.2 DEF: The null set (ϕ), contains no elements: $\phi = \{ \}$

1.3.3 DEF: An n-tuple is an ordered array of n elements, written $\{X_1, X_2, X_3, \ldots, X_n\}$

1.3.4 DEF: The Cartesian product ($A \times B$) OF A and B is the set of all the 2-tuples $\{X_1, X_2\}$ where $X_1 \in A$, $X_2 \in B$:
$A \times B = \{(X_1, X_2) \mid X_1 \in A, X_2 \in B\}$

1.4.1 DEF: A (real-valued) element function f defined on a set S is a rule which associates a real number with every element of the set. The number associated with a particular element is called the value of the function for that element; denoted by $f(w)$ for $w \in S$

1.4.2 DEF: The domain of definition for an element function on a set S is merely the set S itself. (X)
The range of an element function S is the collection of real numbers it associates with the elements of S (Y)

1.4.3 DEF: A class is a set, each of whose elements is a set

1.4.4 DEF: A class $\mathfrak{F}$ is closed with respect to unions and intersections iff $A \cup B \in \mathfrak{F}$ and $A \cap B \in \mathfrak{F}$ for all A & B in $\mathfrak{F}$

1.4.5 DEF: A rule f which associates a real number [$f(A)$] with each $A \in \mathfrak{F}$ is called a set function defined on $\mathfrak{F}$

③ Definitions, Axioms, and Theorems (cont) (pp 26-33)

THEM 2.2.4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

 $\Rightarrow A \cup B = A \cup (\bar{A} \cap B)$

$\therefore \quad P(A \cup B) = P[A \cup (\bar{A} \cap B)]$

$A \cap (\bar{A} \cap B) = \phi \Rightarrow P(A \cup B) = P(A) + P(\bar{A} \cap B)$

$= P(A) + P(B) - P(A \cap B)$ BY THEM 2.2.

2.3.1. DEF.

A single-element event A is a subset of S which has only one element of S belonging to it. ie) if there exists only one $x \in S \ni x \in A \subset S$, then A is a single ele. event

THEM. 2.3.1. Given a sample space S and $A \subset S$

$P(A) = P(A_1) + P(A_2) + \ldots + P(A_K)$ where $A_1, A_2 \ldots A_K$ are single element events and $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_K$

PROOF: $A_i \cap A_j = \phi$ for $i \neq j$

$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_K) = \sum_{i=1}^{k} P(A_i)$ BY AXIOM 3

$\Rightarrow P(A) = \sum_{i=1}^{K} P(A_i)$

THEM 2.3.2. If S has k elements

$$P(A) = n(A)/n(S)$$

satisfies the three axioms for a probability function

1) If S has K elements, then $n(S) = K$

$P(S) = n(S)/n(S) = 1$; satisfying axiom 1

2) If A is a subset of S, it contains a non-negative number of elements: ie) $n(A) \geq 0$ FOR all $A \subset S$

$\therefore \frac{n(A)}{K} = \frac{n(A)}{n(S)} = P(A) \geq 0$ for all $A \subset S$

satisfying axiom 2

3) If $A \cap B = \phi$ then $n(A \cup B) = n(A) + n(B)$

$\Rightarrow \frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} \Rightarrow P(A \cup B) = P(A) + P(B)$

the same reasoning as axiom 3

2.4.1. DEF: If a first operation may be performed n_1 ways, a second n_2 ways, both operations may be performed in $n_1 n_2$ ways

2.4.2 DEF: An arrangement of n ~~symbols~~ elements in a definate order is called a permutation of the n symbols

⑤ Definitions, Axioms, and Theorems (cont.)

2.7.1. DEF: Two events A & B are independent iff

$$P(A \cap B) = P(A)P(B)$$

ALSO: $P(B/A) = P(B)$

2.7.2. A and B are mutually exclusive iff $A \cap B = \phi$

THEM 2.7.1. If $P(A) \neq 0$ and $P(B) \neq 0$, then if A & B are independent, they are not mutually exclusive and if A & B are mutually exclusive, they aren't independent

PROOF: $P(A \cap B) = P(A)P(B) \neq 0$ if A & B are indep.

$\therefore$ they are not mutually exclusive

$A \cap B = \phi$ and $P(A \cap B) = 0$ if A & B are mut. excl.

$P(A)P(B) \neq 0 \Rightarrow$ they aren't independent

2.7.3 DEF: A, B, C are independent iff

1) $P(A \cap B) = P(A)P(B)$ 3) $P(B \cap C) = P(B)P(C)$

2) $P(A \cap C) = P(A)P(C)$ 4) $P(A \cap B \cap C) = P(A)P(B)P(C)$

2.7.4. DEF: An experiment consists of n independent trials iff

1) S is the Cartesian product of n sets $S_1, S_2, \ldots S_n$

2) The probability of occurance of a single-element event $A \subset S$ is the product of the probabilities of occurance of appropriate single element events $A_i \subset S_i$, $i = 1, 2, \ldots, n$

ie) $P(A) = P_1(A_1)P_2(A_2) \cdots P_n(A_n)$ where $A \subset S$, $A_i \subset S_i$, $i = 1, 2, \ldots, n$ and $A, A_1, \ldots, A_n$ are each single element events,

(7) Definitions and Theorems

3.1.1. DEF: A random variable X is a real-valued function of the events of a sample space S.

3.1.2 DEF A random variable X is discrete if its range forms a discrete (countable) set of numbers.
A random variable X is continuous if its range forms a continuous (uncountable) set of real numbers, and the probability of any single value in this range is 0

3.1.3. DEF. The probability function for X is a function $p_X(x)$ of a real variable x is defined to be:

$$p_X(x) = P(X(\omega) = x), \text{ for all real } x$$

3.2.1. DEF. The distribution function for random variable X ($F_X(t)$) is a function of real $t \ni$,

1) The domain of F_X is the whole real line
2) $F_X(t) = P(X \le t)$

NOTE: $P(a < X \le b) = F_X(b) - F_X(a)$

NOTE: $F_Y(t) = \int_{-\infty}^{t} f_Y(y)\,dy$

$\frac{d}{dt} F_Y(t) = f_Y(t)$

$P(a < Y < b) = P(a \le Y \le b) = \int_a^b f_Y(y)\,dy$

DEF: 3.3.1 $E[H(X)] = \sum_{\text{ALL } X} H(x)\,p_X(x)$ IF X IS D.R.V.

$E[H(X)] = \int_{-\infty}^{\infty} H(x) f_X(x)\,dx$ IF X IS C.R.V.

DEF 3.3.2. $E(X) = \mu_X$

DEF 3.3.3 Variance: $\sigma_X^2 = E[(X-\mu_X)^2] = E(X^2) - \mu_X^2$

Standard deviation: $\sigma_X = \sqrt{\sigma_X^2}$

THEM 3.3.1. 1) $E(c) = c$

2) $E(cH(X)) = c\,E(H(X))$

3) $E(H(X) + G(X)) = E[H(X)] + E(G(X)]$

THEM 3.5.1. $Y = a + bX,\ b > 0 \Rightarrow F_Y(t) = F_X\left(\frac{t-a}{b}\right)$

PROOF: $F_Y(t) = P(Y \le t)$

$= P(a + bX \le t)$

$= P\left(X \le \frac{t-a}{b}\right)$

$= F_X\left(\frac{t-a}{b}\right)$

PROBABILITY AND STATISTICS FINAL PLUG SHEET

I) CHAPT. 2: PROBABILITY

1) PERMUTATIONS $_nP_r = \frac{n!}{(n-r)!}$

2) COMBINATIONS $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

3) CONDITIONAL PROBABILITY:

$$P(B/A) = P(B \cap A)/P(A)$$

4) $P(A_j/E) = \frac{P(A_j)P(E/A_j)}{\sum_{i=1}^{k} P(A_i)P(E/A_i)}$ $\quad i = 1, 2, \ldots k$

& A_i FORM A PARTITION OF SAMPLE SPACE

5) INDEPENDENT EVENTS IFF

$$P(A \cap B) = P(A)P(B)$$

II) CHAPT 3: RANDOM VARIABLES & DISTRIBUTION FUNCTIONS

1) $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_Y(y)\,dy$

2) $F_Y(t) = \int_{-\infty}^{t} f_Y(y)\,dy$

3) EXPECTED VALUES

$E[H(x)] = \sum_{\text{ALL } x} H(x)p_X(x)$ FOR DISCRETE $H(x)$

$= \int_{-\infty}^{\infty} H(x)p_X(x)\,dx$ FOR CONTINUOUS $H(x)$

$\mu_X = E(X) =$ MEAN OR AVERAGE

$\sigma_X^2 = E(X^2) - \mu_X^2$

4) $Y = bX + a \Rightarrow F_Y(t) = F_X\left(\frac{t-a}{b}\right)$

5) STANDARD FORM OF A RANDOM VARIABLE

$Z = \frac{(X - \mu_X)}{\sigma_X}$ $\quad \mu_Z = 0;\ \sigma_Z = 1$

III) CHAPT. 4: SOME STANDARD DISTRIBUTIONS

1) BINOMIAL RANDOM VARIABLE $n; p$

$p_X(x) = \binom{n}{x} p^x q^{n-x};\ \mu_X = np;\ \sigma_X^2 = npq$

$F_X(t) = \sum_{k \leq t} \binom{n}{x} p^k q^{n-k}$ (TABLES)

2) GEOMETRIC RANDOM VARIABLE

$p_Y(k) = q^{k-1}p \quad k = 1, 2, 3, \ldots$

$\mu_Y = \frac{1}{p};\ \sigma_Y^2 = q/p^2$

3) HYPERGEOMETRIC RANDOM VARIABLE

$p_Z(k) = \frac{\binom{W}{k}\binom{M-W}{n-k}}{\binom{M}{n}} \quad k = 0, 1, 2, 3, \ldots, n$

$\mu_Z = \frac{nW}{M};\ \sigma_Z^2 = n\frac{W}{M} \cdot \frac{(M-W)}{M} \cdot \frac{(M-n)}{(M-1)}$

4) POISSON RANDOM VARIABLE

$p_X(k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s} \quad k = 0, 1, 2, \ldots$

$\mu_X = \sigma_X^2 = \lambda s$

5) UNIFORM RANDOM VARIABLES

$$f_x(x) = \frac{1}{b-a} \; ; \quad a < x < b$$

$$\mu_x = \tfrac{1}{2}(b+a) \; ; \; \sigma_x^2 = \frac{(b-a)^2}{12}$$

6) EXPODENTIAL RANDOM VARIABLE

$$F_T(s) = (1 - e^{-\lambda s})\mu(s)$$

$$f_T(s) = \lambda e^{-\lambda s}\mu(s)$$

$$\mu_T = \frac{1}{\lambda} \; ; \; \sigma_T^2 = 1/\lambda^2$$

7) NORMAL RANDOM VARIABLE

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$N_z\left(\frac{t-u}{\sigma}\right) \quad n_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$N_z(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

IV) JOINTLY DISTRIBUTED RANDOM VARIABLE

$$p_{x,y}(x,y) = P(X = x, Y = y)$$

$$P(a_1 < X \leq b_1 \,,\, a_2 < Y < b_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_{x,y}(x,y)\,dY\,dX$$

$$F_{XY}(t_1, t_2) = P(X \leq t_1, Y \leq t_2)$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$\sigma_{XY} = E[XY] - \mu_X \mu_Y$$

AUXILIARY PROBLEMS

1. Five politicians meet at a party. How many handshakes are exchanged if each politician shakes hands with every other politician once and only once? $\binom{5}{2}=10$

2. Assuming a year has just 365 days in it, what is the probability no two people in a room containing n persons have the same birthday?

3. An urn contains 3 red balls, 4 white balls, and 5 blue balls. Another urn contains 5 red balls, 6 white balls, and 7 blue balls. One ball is selected from each urn. What is the probability (a) both are white $\frac{1}{9}$ (b) both are the same color? $\frac{5}{72}+\frac{35}{216}+\frac{1}{9}$

4. Two fair dice are tossed. What is the probability that the sum of the dice will be equal to k, for each integer k from 2 to 12 inclusive?

5. Given a group of 4 people find the probability that at least two of them have the same birthday. $1-{}_{365}P_4/365^4$

6. A sample of 3 television sets is selected (without replacement) from a lot of 30 sets. If there are 5 defective sets in the lot, what is the probability the sample contains no defectives? 3 defectives? 1 defective and 2 non-defectives? $\frac{\binom{5}{1}\binom{25}{2}}{\binom{30}{3}}$

7. An electrical circuit consists of 4 switches in series. Assume the operations of the four switches are independent. If for each switch the probability of failure (i.e., remaining open) is .02, what is the probability of circuit failure? .079

8. Rework (7) for the case that the circuit consists of 4 switches in parallel. 8×10^{-8}

9. An electrical assembly consists of two parts connected in series in the order A followed by B. The probability A is defective is .025 and the probability B is defective is .011. What is the probability of having a defective assembly? A non-defective assembly? An assembly that fails only because part B is defective?
DEF .036
WORK .964
.0?06

10. Consider a four engine aircraft (two on each wing) where the probability of an engine failure is .05. Assume that the probability of one engine failing is independent of the behavior of the others. What is the probability of a crash if the engine can fly on any two engines? If the plane requires at least one engine operating on each side in order to remain in the air?

11. Three operators (A,B, and C) alternate in operating a certain machine. The number of parts produced by A, B, and C are in the ratio 3:4:3; and of the parts produced 1% of A's, 2% of B's and 5% of C's are defective. (a) If a part is drawn from the output of their machine, what is the probability it is defective? (b) A part is drawn at random from the machine and is found to be defective. What is the probability A produced it? B produced it? C produced it?

12. Two seeds are chosen from a packet containing seven seeds, two which produce blue flowers, three that produce white, and two that produce red. What is the probability both seeds produce flowers of the same color? (a) Solve by listing sample points (b) Solve by using combinatorial methods. $\frac{10}{42}$

13. How many four-digit numbers can be formed from the digits 5, 6, 7, 8, 9 if each digit can be used only once? 120 If the digits can be repeated? $5^4=625$

14. How many distinguishable ways can the letters in the word "charm" be arranged? $5!=120$ In the word "church"? $6!/2!2!=180$

15. A supermarket is having a sale of unlabeled cans. Included in the sale are 200 cans of corn, 300 cans of beets, and 500 cans of peaches. What is the probability that the first housewife will get a can of vegetables? of corn? Are these two events independent? mutually exclusive? Given that she draws a can of vegetables, what is the probability that it will be corn?

16. Each of two packages of six flashlight batteries contain exactly two inoperable batteries. If two batteries are selected from each package, what is the probability that all four batteries will function?

17. Refer to Problem 16. Suppose that two batteries were randomly selected from package No. 1 and mixed with those in package No. 2. Then two were randomly drawn from the eight in package No. 2. What is the probability that both will function?

18. Madame L. has a jewel box containing four ornate rings. One ring has a diamond and an emerald; another a diamond and a topaz; another an emerald and a topaz; and the other five pearls. Show that the events A: she will select a ring with a diamond, B: she will select a ring with a topaz, C: she will select a ring with an emerald are pairwise independent but not mutually independent.

19. You plan to make two throws of a die. Let X denote the number of times a one will turn up and Y denote the number of times a two will show up.
(a) Describe the probability space on which X and Y are defined.
(b) Construct a table giving the joint probability function of X and Y.
(c) Determine the marginal probability functions of X and Y directly from the original probability space.
(d) Determine the marginal probability functions of X and Y from the joint probability function.
(e) Are X and Y independent?
(f) Find the conditional probability functions of Y for all possible outcomes of X.

60. (Meyer, 5.11) The radient energy E (in BTU/hr/ft^2) is given as the following function of temperature T (in degrees fahrenheit) $E = \frac{.173T^4}{10^8}$. Suppose that the temperature T is a random variable with density function

$$f(t) = \frac{1}{200} t^{-2} \quad \text{for} \quad 40 \leq t \leq 50.$$

Find the density function of the radiant energy E.

61. (Meyer, 5.12) To measure air velocities a tube (known as a Pitot static tube) is used which enables one to measure differential pressure. This differential pressure is given by $P = \frac{1}{2} dV^2$, where d (a constant) is the density of the air and V is the wind speed (mph). If V is a random variable uniformly distributed over $10 \leq V \leq 20$, find the density function of P.

62. (Meyer, 8.22) The probability of a successful rocket launching is .8. Suppose that launching attempts are made until three successful launchings have occured. What is the probability exactly 6 attempts will be necessary? What is the probability fewer than 6 attempts will be required?

63. (Meyer, 8.24) Consider the situation of Problem 62. Suppose that each launching attempt costs \$5000. In addition, a launching failure results in an additional cost \$500. Evaluate the expected cost for the situation described.

64. (Meyer, 8.8) Particles are emitted from a radioactive source. Suppose that the number of such particles emitted during a one-hour period has a Poisson distribution with parameter λ. A counting device is used to record the number of such particles emitted. If more than 30 particles arrive during any one-hour period, the recording device is incapable of keeping track of the excess and simply records 30. If Y is the random variable defined as the number of particles recorded by the counting device, obtain the probability distribution of Y.

65. (Meyer, 8.12) A radioactive source is observed during 7 time intervals each of ten seconds in duration. The number of particles emitted during each period is counted. Suppose that the number of particles emitted, say X, during each observed period has a Poisson distribution with parameter 5.0. (That is, particles are emitted at the rate of .5 particles per second.)

(a) What is the probability that in each of the 7 time intervals, 4 or more particles are emitted?

(b) What is the probability that in at least 1 of the 7 time intervals, 4 or more particles are emitted?

66. (Meyer, 8.13) It has been found that the number of transistor failures on an electronic computer in any one hour period may be considered as a random variable having a Poisson distribution with parameter .1. (That is, one the average, there is one transistor failure every 10 hours). A certain computation requiring 20 hours of computing time is initiated.

(a) Find the probability that the above computation can be successfully completed without a breakdown. (Assume that the machine becomes inoperative only if 3 or more transistors fail.)

(b) Same as (a) except that machine becomes inoperative only if 2 or more transistors fail.

67. (Meyer, 8.14) In forming binary numbers with n digits, the probability an incorrect digit will appear is, say .002. If the errors are independent, what is the probability of finding

(a) zero, (b) one, (c) more than one incorrect digits in a 25 digit binary number? (d) If the computer forms 10^6 such 25 digit numbers per second, what is the probability that an incorrect number is formed during any one second period?

68. (Freund, Prob. 15, page 29) Enumerate the number of ways in which 5 indistinguishable apples can be distributed among 3 children if there is no restriction as to the number of apples any one child can receive. What would be the answer if each child had to receive at least one apple?

69. (Freund, Prob. 16, page 29) Problem 68 deals with a simple problem of occupancy theory arising in physics, namely the problem of determining the number of ways in which r indistinguishable particles can be distributed among n cells, with no restriction as to the number of particles permitted in any one cell. Show that the answer to this general question is $\binom{n+r-1}{r}$ and use this result to verify the answer obtained in Problem 68. (Hint: considering the special case where $r = 5$ and $n = 3$ and using bars to separate the particles in the various cells, p|pp|pp, for example, is an arrangement where the first cell contains 1 particle while the second and third contain 2. Considering all such arrangements of p's and bars, the result follows immediately.)

70. (Freund, Prob. 17, page 30) <u>Occupancy theory</u>. Find (a) the number of ways in which r distinguishable particles can be distributed among n cells with no restriction as to the number of particles allowed in any one cell, and (b) the number of ways in which r indistinguishable particles can be distributed among n cells with at least 1 particle in each cell. [For part (b) assume that $r \geq n$.]

71. (Freund, Prob. 9, page 151) If z is a random variable having the standard normal distribution, use Table III to find

(a) the probability that z assumes a value greater than 2.68

(b) the probability that z assumes a value less than 1.73

(c) the probability that z assumes a value greater than -0.66

(d) the probability that z assumes a value less than -1.88

(e) the probability that z assumes a value between -1.05 and -1.65

(f) the probability that z assumes a value between -0.05 and 1.05.

72. (Freund, Prob. 10, page 152) If z is a random variable having the standard normal distribution, use Table III to find z so that

(a) the probability that z assumes a value between 0 and z is 0.4515

(b) the probability that z assumes a value greater than z is 0.3121

(c) the probability that z assumes a value less than z is 0.4562

(d) the probability that z assumes a value between $-z$ and z is 0.7416

73. (Freund, Prob. 13, page 152) The density of certain glass bricks is a random variable having a normal distribution with $\mu = 2.480$ and $\sigma = 0.03$. Below what density can we expect to find the lightest 20 per cent of these bricks?

74. (Freund, Prob. 14, page 152) If one ball bearing is selected at random from a very large shipment, its diameter is a random variable whose distribution is normal with $\mu = 0.397$ in. and $\sigma = 0.005$ in. What is the probability that the diameter of a ball bearing thus selected will exceed 0.400 in.?

75. (Freund, Prob. 15, page 152) The lifetime of a certain kind of battery is a random variable having a normal distribution with $\mu = 300$ hours and $\sigma = 35$ hours. Find the probability that one of these batteries will have a lifetime of more than 320 hours. Also find the value above which we can expect to find the best 25 per cent of these batteries.

76. (Freund, Prob. 3, page 141) Given the joint density

$$f(x_1,x_2) = \begin{cases} \dfrac{k}{(1 + x_1 + x_2)^3} & \text{for } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find k and the marginal densities of X_1 and X_2. Also find the conditional density of X_1 given that X_2 assumes the value x_2.

77. (Freund, Prob. 8, page 142) (a) Given $f(x,y) = \frac{1}{81} x^2 y^2$ for $0 < x < 3$ and $0 < y < 3$, and $f(x,y) = 0$ elsewhere, check whether x and y are independent.

(b) Given $f(x,y) = \frac{2}{81} x^2 y^2$ for $0 < x < y$ and $0 < y < 3$, and $f(x,y) = 0$ elsewhere, check whether x and y are independent.

78. (Freund, Prob. 9, page 142) Suppose that the price of a certain item (in dollars), which depends on the cost of production and prices of raw materials, may be looked upon as a random variable having the uniform density

$$f(p) = \begin{cases} 5 & \text{for } 0.20 < p < 0.40 \\ 0 & \text{elsewhere} \end{cases}$$

For a fixed price p, total sales (in 10,000 units) may be looked upon as a random variable having the conditional exponential distribution

$$f(s|p) = \begin{cases} p \cdot e^{-ps} & \text{for } s > 0 \\ 0 & \text{elsewhere} \end{cases}$$

What is the probability that sales of this item will exceed 25,000 units?

79. (Freund, Prob. 1, page 88) Given a joint probability distribution whose values $f(x,y)$ are

$f(1,1) = 5/27$ $\quad f(1,2) = 1/27$ $\quad f(1,3) = 3/27$

$f(2,1) = 4/27$ $\quad f(2,2) = 3/27$ $\quad f(2,3) = 4/27$

$f(3,1) = 2/27$ $\quad f(3,2) = 3/27$ $\quad f(3,3) = 2/27$

find

(a) the marginal distribution of X

(b) the marginal distribution of Y

(c) the conditional distribution of X given that the value of Y is 2

(d) the conditional distribution of Y given that the value of X is 1

80. (Freund, Prob. 4, page 88) Check whether the two random variables of Problem 79 are independent.

81. (Meyer, Prob. 7.39) The following example illustrates that $\rho = 0$ does not imply independence. Suppose that (X,Y) has a joint probability distribution given by the table:

Y \ X	-1	0	1
-1	$\frac{1}{8}$	$\boxed{\frac{1}{8}}$	$\frac{1}{8}$
0	$\boxed{\frac{1}{8}}$	0	$\boxed{\frac{1}{8}}$
1	$\frac{1}{8}$	$\boxed{\frac{1}{8}}$	$\frac{1}{8}$

(a) Show that $E(XY) = E(X)E(Y)$ and hence $\rho = 0$.

(b) Indicate why X and Y are not independent.

(c) Show that this example may be generalized as follows. The choice of the number $\frac{1}{8}$ is not crucial. What is important is that all the circled values are the same, all the boxed values are the same, and the center value equals zero.

20. A coin is to be tossed. If heads comes up, a die will then be tossed; while if tails comes up, the coin will be tossed again. Assign the number 1 to heads and the number 0 to tails, and let X denote the number to come up on the first toss and Y denote the number to come up on the second toss. Repeat Parts (a) through (f) of Problem 19.

21. An equipment salesman can contact either one or two customers per day with probability 1/3 or 2/3, respectively. Each contact will result in either no sale or a $50,000 sale with probability 9/10 and 1/10, respectively. What is the expected value of the daily sales?
$8333

22. Suppose that weather records show that on the average 3 of the 30 days in November are rainy days. (a) Assuming a binomial distribution with each day of November as an independent trial, find the probability that next November will have at most 2 rainy days. (b) Give reasons why you may not be justified in using the binomial distribution in solving (a).

23. Use the Poisson approximation to the binomial distribution to solve 22 (a).

24. Use the Poisson approximation to the binomial distribution to calculate the probability that at most 1 person in a group of 500 will have a birthday on Christmas. Assume 365 days in the year.

25. If one is given a set of data $X_1, X_2, \ldots, X_n$, the mean $\overline{X}$ and the variance S^2 for the data are defined by

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad \text{and} \qquad S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$$

Compare these definitions with the definitions of the mean and variance of the discrete random variable and prove that

$$S^2 = \frac{\sum_{i=1}^{n} X_i^2}{n} - \overline{X}^2.$$

Then find the mean and variance for the following data which represent final averages for 5 students in a certain course: 93, 78, 70, 60, 45.

26. The correlation coefficient for two random variables X and Y is defined to be

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

a) Find the correlation coefficient in Problem 1 on Page 113 of Freund. (b) Find the correlation coefficient for the example on page 113 of Freund's book.

27. The correlation coefficient between the X's and Y's in a set of data consisting of ordered pairs $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is defined to be

$$r = \frac{\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X})^2} \sqrt{\sum_{i=1}^{n} (Y_i - \overline{Y})^2}}$$

Compare this definition with the definition given in Problem 26 and prove that

$$r = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{X}\overline{Y}}{\sqrt{\sum_{i=1}^{n} X_i^2 - n\overline{X}^2} \sqrt{\sum_{i=1}^{n} Y_i^2 - n\overline{Y}^2}}$$

Then find the correlation coefficient for the following data which represent the entrance examination scores and final averages of five students in a given freshman mathematics course:

Student	entrance test score	final course average
1	30	90
2	20	70
3	25	75
4	35	85
5	15	60

28. If $f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ is the density function of a continuous random variable X, find the number a such that $Pr(X>a) = Pr(X<a)$. Also, find the number b such that

$Pr(X>b) = .05$.

29. Suppose the life in hours of a certain kind of radio tube has the density

$$f(x) = \begin{cases} 100/x^2 & \text{if } x \ge 100 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the probability that none of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation? 8/27

(b) What is the probability all three of the original tubes will have been replaced during the first 150 hours of operation? 1/27

(c) In this model what is the mean life of these tubes? What is your reaction to this result?

30. Given $f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$, find the density functions of (a) $Y = 1/X$ and (b) $Y = \log_e X$

31. A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 ft. of the center of the track, the track will be sufficiently damaged to disrupt traffic. With a certain bombsight the density of points of impact of a bomb is

$$f(x) = \begin{cases} (100+x)/10000 & \text{for } -100<x<0 \\ (100-x)/10000 & \text{for } 0<x<100 \\ 0 & \text{otherwise} \end{cases}$$

where x represents the deviation from the aiming point, which is the center of the track in this case. If all three bombs are used, what is the probability that the track will be damaged? $1-(.36)^3$

32. Referring to the above problem, the planes can carry eight bombs of smaller size but one of these must hit within 15 ft. of the center of the track to damage it. Should the lighter or heavier bombs be used on this mission? $1-(.7225)^{10}$

33. Rework the above two problems assuming that the density function for X is normal with the same mean and variance as that of the density function given in Problem 31.

34. A country filling station is supplied with gasoline once a week. If its weekly volume X of sales in thousands of gallons had density function

$$f(x) = \begin{cases} 5(1-x)^4 & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given week shall be .01? $1-\sqrt[5]{.01}$

35. Work the above problem assuming that X has a normal distribution with the same mean and variance as that of the given density function.

36. Suppose you plan to choose a number X at random between 0 and 1 and then to draw a chord at distance X from the center of a circle of radius 1. What is the probability that the length of the chord will be less than the side of an inscribed equilateral triangle?

37. Suppose you plan to choose at random a number Y between 0 and π and then to draw a chord having a central angle Y in a circle. What is the probability that the length of the chord will be less than the side of an inscribed equilateral triangle?

38. A college professor in a university much larger than Rose Polytechnic Institute has a course with a thousand students in it (It can happen!). He gives the students an examination and obtains an average score of 60 (out of 100) with a standard deviation of 13. He assumes (since he has so many scores) that the scores are normally distributed with mean $\mu = 60$ and $\sigma = 13$. Under this assumption:

(a) What is the probability a student gets a score of more than 100? Comment on your answer.

(b) At least what score should a student have to get an A on the test if the professor decides he will give 5% of the students A's?

(c) If the professor decides to fail a student if he gets a score below 45, how many of the 1000 students would be expected to fail?

39. (Meyer, 1.15) A certain type of electric motor fails either by seizure of the bearings, or by burning out of the electrical windings, or by wearing out of the brushes. Suppose that seizure is twice as likely as burning out, which is four times as likely as brush wearout. What is the probability failure will be by each of these mechanisms?

BURN 4/13
BRUSH 1/13
SEIZ 8/13

40. (Meyer, 2.6) A lot consists of 10 ~~good~~ articles, 4 with minor defects, and 2 with major defects. One article is chosen at random. Find the probability that

(a) it has no defects. $\frac{4}{10}$

(b) it has no major defects. $\frac{8}{10}$

(c) it is either good or has major defects. $\frac{6}{10}$

41. (Meyer, 2.7) If from the lot of articles described in Problem 40, two articles are chosen without replacement, find the probability that

(a) both are good $\frac{2}{15}$

(b) both have major defects $\frac{1}{45}$

(c) at least one is good $\frac{4}{5}$

(d) at most one is good $\frac{39}{45}$

(e) exactly one is good $\frac{8}{15}$

(f) neither has major defects $\frac{28}{45}$

(g) neither is good $\frac{1}{3}$

42. (Meyer, 3.9) In a bolt factory, machines A, B, and C manufacture 25, 35, and 40% of the total output, respectively. Of their outputs 5, 4, and 2%, respectively, are defective bolts. A bolt is chosen at random and found to be defective. What is the probability the bolt came from Machine A? B? C?

43. (Meyer, 3.10) Let A and B be two events associated with an experiment. Suppose $P(A) = .4$ while $P(A \cup B) = .7$, let $P(B) = p$.

(a) For what choice of p are A and B mutually exclusive?

(b) For what choice of p are A and B independent?

44. (Meyer, 3.13) A binary number is one composed only of the digits zero and one. (For example, 1011, 1100, etc.) These numbers play an important role in the use of electronic computers. Suppose that a binary number is made up of n digits. Suppose the probability of an incorrect digit appearing is p and that errors in different digits are independent of one another. What is the probability of forming an incorrect number?

45. Four radio signals are emitted successively. If the reception of any one signal is independent of the reception of another and if these probabilities are .1, .2, .3, and .4, respectively, compute the probability G signals will be received for $G = 0, 1, 2, 3, 4$.

46. (Meyer, 4.3) Suppose that the random variable X has possible values 1, 2, 3, ... and $P(X = j) = (1/2)^j$ for $j = 1,2,3,\ldots$.

(a) Compute $P(X \text{ is even})$.

(b) Compute $P(X \geq 5)$.

(c) Compute $P(X \text{ is divisible by } 3)$.

47. (Meyer 4.6) Rockets are launched until the first successful launching has taken place. If this does not occur within 5 attempts, the experiment is halted and the equipment inspected. Suppose that there is a constant probability of .8 of having a successful launching and that successive attempts are independent. Assume the cost of the first launching is K dollars while subsequent launchings cost $K/3$ dollars. Whenever a successful launching takes place, a certain amount of information is obtained which may be expressed as financial gain of C dollars. If T is the net cost of this experiment, find the probability distribution of T

48. (Meyer, 4.14) The percentage of alcohol (100 X) in a certain chemical may be considered as a random variable, where X, $0 < X < 1$ has the following density function

$$f(x) = 20x^3 (1 - x) \quad \text{For } 0 < x < 1$$

(a) Obtain an expression for the cumulative distribution function $F(x)$ and sketch its graph. $5x^4 - 4x^5$

(b) Find $P(X \leq 2/3)$. $5(\frac{2}{3})^4 - 4(\frac{2}{3})^5$

(c) Suppose that the selling price of the above chemical depends on the alcohol content. Specifically, if $1/3 < X < 2/3$, the chemical sells for C_1 dollars per gallon; otherwise it sells for C_2 dollars per gallon. If the cost is C_3 dollars per gallon, find the probability function of the net profit per gallon. $E(x) = \frac{C_1 + 2C_2 - 3C_3}{3}$

49. (Meyer, 4.16) The diameter of an electric cable, say X, is assumed to be a continuous random variable with density function

$$f(x) = 6x(1 - x) \quad \text{for} \quad 0 < x < 1$$

(a) Check that $f(x)$ is a good density function and sketch it.

(b) Obtain an expression for the cdf of X and sketch it. $3x^2-2x^3$

(c) Determine a number b such that

$$P(X < b) = 2P(X > b).$$

(d) Compute $P(X \leq 1/2 \mid 1/3 < X < 2/3)$.

50. (Meyer, 4.18) Let X be the life length of an electronic device (measured in hours). Suppose that X is a continuous random variable with density function $f(x) = G/x^n$ for $2000 < x < 10{,}000$.

(a) For $n = 2$ determine G.

(b) For $n = 3$ determine G.

(c) For general n determine G.

(d) What is the probability the device will fail before 5000 hours have elapsed?

(e) Sketch the cdf for (c) and determine its algebraic form.

51. (Meyer, 4.25) Suppose that the life length (in hours) of a certain radio tube is a continuous random variable X with density function $f(x) = 100/x^2$ for $x > 100$.

(a) What is the probability that a tube will last less than 200 hours if it is known that the tube is still functioning after 150 hours of service?

(b) What is the probability that if 3 such tubes are installed in a set, exactly one will have to be replaced after 150 hours of service?

(c) What is the maximum number of tubes that may be inserted into a set so that there is a probability of .5 that after 150 hours of service all of them are still functioning?

52. (Meyer, 7.1) Find the expected value of the following random variables:

(a) The random variable T defined in Problem 47.

(b) The random variable X defined in Problem 50.

53. (Meyer, 7.2) Show that $E(X)$ does not exist for the random variable X defined in Problem 51.

54. (Meyer, 7.4) In the manufacture of petroleum, the distilling temperature, say T^{o} (degrees centigrade) is crucial in determining the quality of the final product. Suppose that T is considered as a random variable uniformly distributed over the interval $150 \leq T \leq 300$. Suppose that it costs C_1 dollars to produce one gallon of petroleum. If the oil distills at a temperature less than $200^{o}C$, the product is known as naptha and sells for C_2 dollars per gallon. If it is distilled at a temperature greater than $200^{o}C$, it is known as refined oil distillate and sells for C_3 dollars per gallon. Find the expected net profit per gallon.

55. (Meyer 7.5) A certain alloy is formed by combining the melted mixture of two metals. The resulting alloy contains a certain percent of lead, say X, which may be considered a random variable with density function. $f(x) = 3/5 \cdot 10^{-5}\, x(100 - x)$ for $0 \leq x \leq 100$. Suppose that P, the net profit realized in selling this alloy (per pound) is the function of the percent content of lead $P = C_1 + C_2 X$. Compute the expected profit per pound.

56. (Meyer, 7.6) Suppose that an electronic device has a life length X (in units of 1000 hours) which is considered as a random variable with density function $f(x) = e^{-x}$ for $x > 0$. Suppose the cost of manufacturing one such item is \$2.00. The manufacturer sells the item for \$5.00 but guarantees a total refund if $X \leq .9$. What is the manufacturer's expected profit per item?

57. (Meyer, 7.9) A lot of 10 electric motors must either be totally rejected or sold depending on the outcome of the following procedure: Two motors are chosen at random and inspected. If one or more is defective, the lot is rejected. Otherwise it is accepted. Suppose each motor costs \$75 and is sold for \$100. If the lot contains 1 defective motor what is the manufacturer's expected profit?

58. (Meyer, 5.8) A fluctuating electric current I may be considered as a uniformly distributed random variable over the interval $9 \leq I \leq 11$. If this current flows through a 2-ohm resistor, find the density function of the power $P = 2I^2$.

59. (Meyer 5.9) The speed of a molecule in a uniform gas at equilibrium is a random variable V with density function $f(v) = av^2 e^{-bv^2}$ for $v > 0$ where $b = \frac{m}{2GT}$ and G, T, and m denote Boltzman's constant, the absolute temperature and the mass of the molecule, respectively.

(a) Evaluate the constant a (in terms of b)[Hint: Use the fact

$$\int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$ and integrate by parts.]

(b) Derive the density function of the kinetic energy of the molecule $W = \frac{mV^2}{2}$.

ROBERT J. MARKS
10-6-70

49

1) A = 3 DEF) $P(A) = 3/8$
B 5 NON-DEF) $P(B) = 5/8$

a) $$\frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{3\ \frac{5\cdot 4}{2}}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{30}{56}\ \frac{3\ \frac{5\cdot 4}{2}}{\frac{2\cdot 8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{6}{14}$$

b) $$\frac{\binom{5}{2}\binom{6}{2}}{\binom{8}{4}} = \frac{5\cdot 2\cdot 15}{\frac{2\cdot 8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{\frac{2\cdot 5\cdot 4}{2}\ \frac{3\cdot 6\cdot 5}{2}}{\frac{2\cdot 8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} =$$

c) $$\frac{\binom{3}{2}\binom{6}{2}}{\binom{8}{4}} = \frac{3\ \frac{3\cdot 6\cdot 5}{2}}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{36}{56}$$

-15

2) P(FAIL) = P(switch 1 fails and (switch two or switch 3 fail))

$= P[1 \cap (2 \cup 3)]$

OR → $\cup$
AND → $\cap$

~~$P(\text{WORK}) = P[\bar{1} \cup (\overline{2 \cup 3})]$~~

~~1, 2, 3 are indep → $\bar{1}, \bar{2}, \bar{3}$ are indep~~

~~$P(\text{WORK}) = P(\bar{1})P(\overline{2 \cup 3})$~~

~~$P(\overline{2 \cup 3}) =$~~

-12

~~$P(\text{WORK}) = P[\bar{1} \cap (\overline{2 \cup 3})]$~~

~~$P(\bar{1} \cap (\overline{2 \cup 3})) = P(\overline{2 \cup 3}) - P[1 \cap (\overline{2 \cup 3})]$~~

$P(\text{FAIL}) = P[1 \cap (2 \cup 3)]$

$= P[\overline{\bar{1} \cup (\overline{2 \cup 3})}]$

$= P[\overline{1 \cup (\overline{2 \cup 3})}]$

$= P(1) P(\overline{2 \cup 3})$

$A \cap B = \overline{\bar{B} \cup}$

$A' \cap B'$

$P(2 \cup 3) = P(2)P(3) = .01 \Rightarrow P(\overline{2 \cup 3}) = .99$

$\Rightarrow P(\text{FAIL}) = (.1)(.99) = .099$

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(a) Evaluate the constant a (in terms of b)[Hint: Use the fact

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ROBERT J. MARKS
10-6-70

49

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B 5 NON-DEF) $P(B) = 5/8$

a) $$\frac{\binom{3}{2}\binom{5}{2}}{\binom{8}{4}} = \frac{3\ \frac{5\cdot 4}{2}}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{3\ \frac{5\cdot 4}{2}}{\frac{28\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{6}{14}$$

b) $$\frac{\binom{5}{2}\binom{6}{2}}{\binom{8}{4}} = \frac{5\cdot 2\cdot 15}{\frac{28\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{\frac{2\cdot 5\cdot 4}{2}\ \frac{36\cdot 5}{2}}{\frac{28\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} =$$

c) $$\frac{\binom{3}{2}\binom{6}{2}}{\binom{8}{4}} = \frac{3\ \frac{36\cdot 5}{2}}{\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{36}{56}$$

-15

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-12

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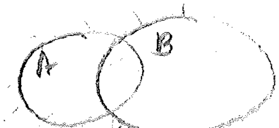
$= P[\overline{\bar{1} \cup \overline{(2 \cup 3)}}]$

$= P(\bar{1})P(\overline{2 \cup 3})$



$A \cap B = \overline{\bar{B} \cup}$

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$P(2 \cup 3) = P(2)P(3) = .01 \Rightarrow P(\overline{2 \cup 3}) = .99$

$\Rightarrow P(\text{FAIL}) = (.1)(.99) = .099$

3) $\dfrac{P^{365}_{3}}{(365)^3} = \dfrac{365\cdot 364\cdot 363}{365^3}$ -10

$= \dfrac{364\cdot 363}{365^2}$

$= \dfrac{129732}{133225}$

$$\begin{array}{r} 364 \\ 363 \\ \hline 1092 \\ 19840 \\ 109200 \\ \hline 129732 \end{array} \qquad \begin{array}{r} 365 \\ 365 \\ \hline 1825 \\ 21900 \\ 109500 \\ \hline 133225 \end{array}$$

4) $P(A) = 2/10$ $D \Rightarrow$ DEFECTIVE

$P(B) = 3/10$

$P(C) = 5/10$

$P(D|A) = 3/100$ $P(D/A) = \dfrac{P(D\cap A)}{P(A)}$

$P(D|B) = 2/100$

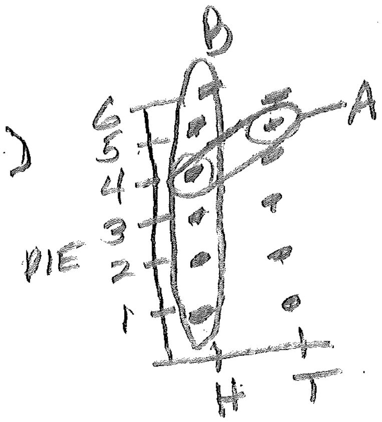
$P(D|C) = 1/100$

P($P(E/D) = \dfrac{P(D)\,P(D/E)}{\sum P(E)\,P(D/E_n)}$

$P(D/A) = \dfrac{P(D\cap A)}{P(A)}$

-13

5) a)



$S = \{(X_1, X_2) \mid X_1 = 1,2,3,4,5,6 \quad X_2 = \overset{1}{H}, \overset{0}{T}\}$

a) if indep $P(A \cup B) = 0$

$P(A \cup B) = P(A)P(B)$

$\frac{1}{12} = \frac{2}{12} \cdot \frac{1}{2}$

∴ is independent

b) Not mutually exclusive,
for no independent event
can be mut. exclusive if $P(A) \neq 0$
and $P(B) \neq 0$

$-2 \mid$

BOB MARKS

Probability – Test 1

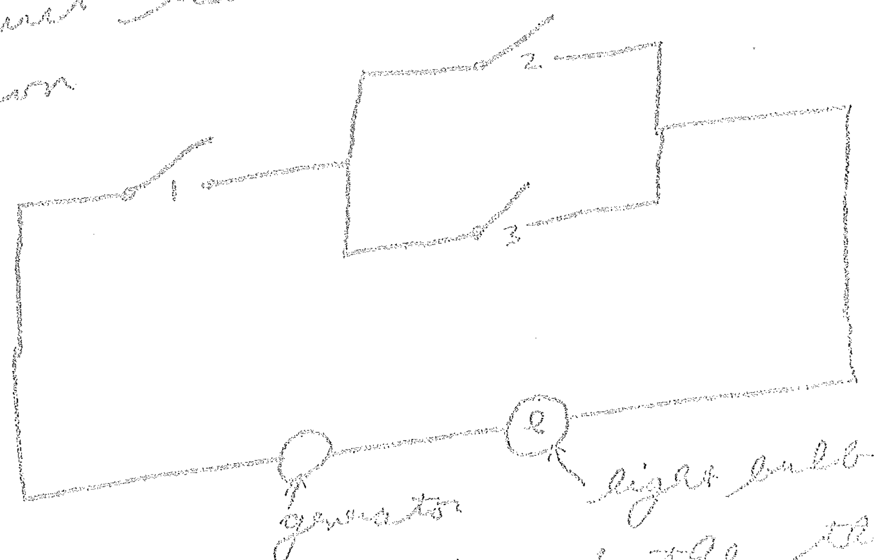
(30) 1. A lot contains 8 articles, 3 of which are defective and 5 of which are non-defective. Four articles are drawn at random without replacement from the lot. What is the probab[ility]

(a) exactly two of them are defective?

(b) at most two of them are defective?

(c) at least two of them are defective?

(15) 2. A circuit has three switches connect[ed] as shown



Assuming the operations of the three switches are independent and for each switch the probability of failure (i.e., remaining open) is .1, what is the probability of no current through the light bulb?

(15) 3. Three people are in a room. Compute the probability at least two of them [illegible] the same season of the

Bob Marks

(20) 4. During a typical day in a bolt factory, Machines A, B and C manufacture 20, 30, and 50% of the total output, respectively. Of their outputs 3%, 2%, and 1% respectively are defective bolts. A bolt is drawn at random at the end of the day.

(a) What is the probability it is defective?

(b) Given it is defective, what is the probability it came from Machine A?

(20) 5. A perfectly balanced six-sided die with sides numbered 1 through 6 and a perfectly fair coin are tossed together.

(a) Write down the sample space for this experiment.

(b) Let A be the event that the number on the die plus the number of heads totals five and B be the event a head appears on the coin.

(i) Are A and B independent? Show why or why not.

(ii) Are A and B mutually exclusive? Show why or why not.

1. An urn contains three balls numbered 2, 3, and 4, respectively. Two balls are drawn from the urn **with** replacement. Let X be the sum of the two numbers that occur. Find the probability function of X.

2. A point is chosen at random between 0 and 2 on the x-axis in the x-y plane. A circle centered at the origin is then drawn in the plane with radius determined by the given point. Compute the probability the area of the circle is between $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

3. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} k(2x - x^2) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

(a) Show that k must be $3/4$.

(b) Find $P\left(\frac{1}{2} < X < 1\right)$.

(c) Find the cumulative distribution function of X.

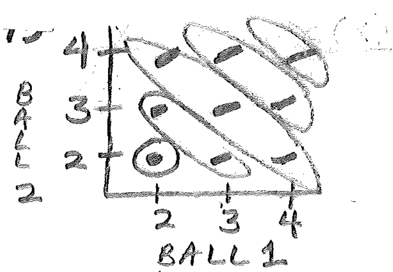
(d) Find μ_X and σ_X^2.

(e) Find the cumulative distribution function of $Y = X^3$.

(f) Find the density function of $Y = X^3$.

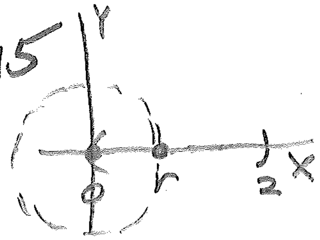
ROBERT J. MARKS

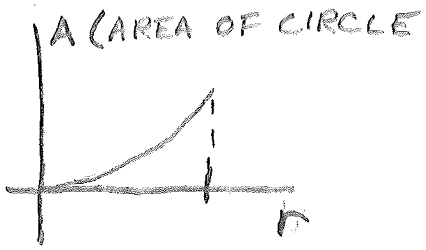
95/100

1)10 

x = sum of two balls

x	$p(x)$
4	1/9
5	2/9
6	3/9
7	2/9
8	1/9

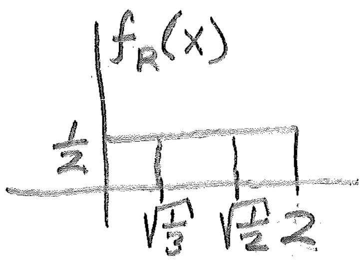
2)15 



$$\pi r^2 = \frac{\pi}{3} \Rightarrow r = \sqrt{\tfrac{1}{3}}$$

$$\pi r^2 = \frac{\pi}{2} \Rightarrow r = \sqrt{1/2}$$

p(AREA BETWEEN $\frac{\pi}{3}$ & $\frac{\pi}{2}$) = p(RADIUS BETWEEN $\sqrt{\frac{1}{2}}$ & $\sqrt{\frac{1}{3}}$)



$$p\left(\sqrt{\tfrac{1}{3}} \le r \le \sqrt{\tfrac{1}{2}}\right) = \int_{\sqrt{\frac{1}{3}}}^{\sqrt{\frac{1}{2}}} \frac{1}{2}\,dx$$

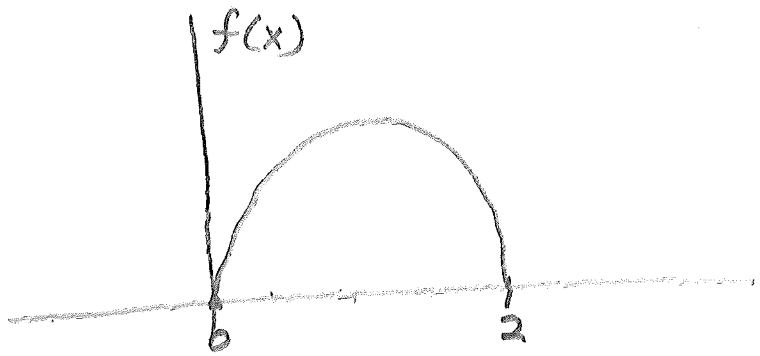
$$\frac{1}{2}\left(\sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{3}}\right)$$

$$p\left(\frac{\pi}{3} \le A \le \frac{\pi}{2}\right) = \frac{1}{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{3}\right]$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{3}}{6}$$

3)



10 a) $\int_0^2 (k2x - kx^2) = 1$

$kx^2 - \frac{kx^3}{3}\Big]_0^2 = 1$

$4k - \frac{8k}{3} = 1 = k(4 - \frac{8}{3}) = \frac{4}{3}k \Rightarrow k = 3/4$

10 b) $P(\frac{1}{2} < X < 1) = \frac{3}{4}\int_{\frac{1}{2}}^{1} (2x - x^2)\,dx$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_{\frac{1}{2}}^{1}$ $\frac{6}{24} - \frac{1}{24} = \frac{5}{24}$

$= \frac{3}{4}\left[(1 - \frac{1}{3}) - (\frac{1}{4} - \frac{1}{24})\right]$

$= \frac{3}{4}\left[\frac{2}{3} - \frac{5}{24}\right] = \frac{3}{4}\left[\frac{16}{24} - \frac{5}{24}\right] = \frac{\cancel{3}}{4}\left(\frac{11}{\cancel{24}\,8}\right) = \frac{11}{32}$

$(t = x)$

10 c) $F_x(x) = \frac{3}{4}\int_0^t (2x - x^2)\,dx$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^t = \frac{3}{4}\left[t^2 - \frac{t^3}{3}\right]$

$\therefore F_x(x) = \frac{3}{4}(x^2 - \frac{x^3}{3})$ $0 \le x \le 2$

$= 1$ $x \ge 2$

$= 0$ $x \le 0$

10 d) $E(x^2) = \frac{3}{4}\int_0^2 x^2(2x - x^2)\,dx$

$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$

$= \frac{3}{4}\left[\frac{1}{2}x^4 - \frac{x^5}{5}\right]_0^2$

$= \frac{3}{4}\left[\frac{1}{2}16 - \frac{32}{5}\right]$

$$8 \quad \mu_X = E(X) = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{64-48}{12}\right] = \frac{3}{4}\,\frac{16}{12} = 1$$

$$12 \quad \sigma_X^2 = E(X^2) - \mu_X^2$$

$$= \frac{6}{5} - 1 = \frac{1}{5}$$

e) 12
$$F_X(Y) = P(Y \leq x)$$

$$= P(X^3 \leq x)$$

$$= P(X \leq \sqrt[3]{x})$$

$$= F_X(\sqrt[3]{x})$$

$$F_X(Y) = \frac{3}{4}\left(x^{\frac{2}{3}} - \frac{x}{3}\right) \qquad 0 \leq x^{\frac{1}{3}} \leq 2$$

$$= \frac{3}{4}\left(x^{\frac{2}{3}} - \frac{x}{3}\right) \qquad 0 < x < 8$$

$$\Rightarrow F_X(Y) = 0 \qquad x \leq 0$$

$$= \frac{3}{4}\left(x^{\frac{2}{3}} - \frac{x}{3}\right) \qquad 0 \leq x \leq 8$$

$$= 1 \qquad x > 8$$

f) 8
$$f_X(Y) = \frac{d}{dx}F_X(Y)$$

$$= 0 \qquad x \leq 0 \qquad x \geq 8$$

$$= \frac{3}{4}\left[\frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}\right] \qquad 0 \leq x \leq 8$$

$$f_X(Y) = \frac{1}{2}x^{-\frac{1}{3}} - \frac{1}{4} \longleftarrow \text{This is } \cancel{\text{not}} \text{ correct} \qquad 0 \leq x \leq 8$$

$$= 0 \qquad \text{OTHERWISE}$$

$$f_X(Y) = \frac{3}{4}\left(2x^{\frac{1}{3}} - x^{\frac{2}{3}}\right) \qquad 0 \leq x \leq 8$$

$$= 0 \qquad \text{OTHERWISE}$$

BOB MARKS

Test 1 Probability

(20) 1. Prove that the mean and variance of the Poisson random variable with parameter α are both equal to α.

(20) 2. A lot contains 10 objects, 3 of which are defective. A random sample of 4 is taken from this lot. What is the probability there are at least two defective items in the sample assuming

(a) the sample was taken <u>with</u> replacement?

(b) the sample was taken <u>without</u> replacement?

(20) 3. It has been observed that cars passing a certain point on a rural road form a Poisson process with an average rate of 3 per hour. Let X be the number which pass this point in a 40 minute interval and let T be the time (in hours) one has to wait until he sees the first car. Find

(a) $P(X \geq 2)$ and (b) $P(T \leq \frac{1}{2})$.

(25) 4. The lifetime of a certain kind of battery is approximately a normal random variable with $\mu = 300$ hours and $\sigma = 20$ hours.

(a) Find the probability that one of these batteries will have a lifetime between 310 and 330 hours.

(b) Find the value above which we can expect to find the best 10% of these batteries.

... with <u>replacement</u> from an

BOB MARKS

86

(1) $P_x(x) = \dfrac{\alpha^x e^{-\alpha}}{x!}$

a) $\mu_x = E(x) = \sum_{x=0}^{\infty} x \dfrac{\alpha^x e^{-\alpha}}{x!}$

$= \sum_{x=1}^{\infty} x \dfrac{\alpha^x e^{-\alpha}}{x!} = \sum_{x=1}^{\infty} \dfrac{\alpha^x e^{-\alpha}}{(x-1)!} = \sum_{x=1}^{\infty} \dfrac{\alpha \cdot \alpha^{x-1} e^{-\alpha}}{(x-1)!}$

$= \alpha \sum_{x=1}^{\infty} \dfrac{\alpha^{x-1} e^{-\alpha}}{(x-1)!}$

LET $Y = X-1 \Rightarrow E(x) = \mu(x) = \alpha \sum_{Y=0}^{\infty} \dfrac{\alpha^Y e^{-\alpha}}{Y!}$

SINCE $\sum_{Y=0}^{\infty} \dfrac{\alpha^Y e^{-\alpha}}{Y!} = 1$; $\mu(x) = \alpha$

b) $\sigma_x^2 = E(x^2) - \mu_x^2$

$= E[x(x-1)] + E(x) - \mu_x^2$

$= E[x(x-1)] + \alpha - \alpha^2$

$E[x(x-1)] = \sum_{x=0}^{\infty} x(x-1) \dfrac{\alpha^x e^{-\alpha}}{x!}$

$= \sum_{x=2}^{\infty} x(x-1) \dfrac{\alpha^x e^{-\alpha}}{x!}$

$= \sum_{x=2}^{\infty} \dfrac{(\alpha^2 \cdot \alpha^{x-2}) e^{-\alpha}}{(x-2)!} = \alpha^2 \sum_{x=2}^{\infty} \dfrac{\alpha^{x-2} e^{-\alpha}}{(x-2)!}$

$z = x-2 \Rightarrow E[x(x-1)] = \alpha^2 \sum_{z=0}^{\infty} \dfrac{\alpha^z e^{-\alpha}}{z!} = \alpha^2$

$\therefore \sigma_x^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$

(2) b) $M = 10$ $W = 3$ $n = 4$

$P_x(k) = \dfrac{\binom{W}{x}\binom{M-W}{n-x}}{\binom{M}{n}}$

$P(x \geq 2) = \sum_{x=2}^{4} \dfrac{\binom{3}{x}\binom{7}{4-x}}{\binom{10}{4}}$

-3

$= \dfrac{3 \cdot \cancel{21}}{\frac{\cancel{3} \cdot 10 \cdot \cancel{9} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot 3 \cdot 2 \cdot 1}} +$

$3 \quad \dfrac{7 \cdot 6}{2 \cdot 1}$

$\ldots = \dfrac{3}{\ldots} + \dfrac{1}{\ldots} = \dfrac{4}{\ldots}$

(2)-CONT)

a) $P_X(x) = \binom{n}{x} p^x q^{n-x}$

$P(X \geq 2) = \sum_{n=2}^{4} \binom{4}{x} \left(\frac{3}{10}\right)^x \left(\frac{7}{10}\right)^{4-x}$

$(P(x) \geq 2) = P(X \leq 1)$ $= \binom{4}{1}\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)^3$

$= \frac{12}{10^4} \cdot 7^3$

$= .4116$

$\begin{array}{r} 49 \\ 7 \\ \hline 343 \\ 12 \\ \hline 686 \\ 3430 \\ \hline 4116 \end{array}$

-4

~~3) $\lambda = \frac{3 \text{ CARS}}{HR}$ $s = \frac{4}{6} HR \Rightarrow \lambda s = 2$ CARS~~

~~$P_X(x) = \frac{(\lambda s^x)}{x!} e^{-\lambda s}$~~

~~$P(X = 40) = p_X(x) =$~~ $\frac{2^x e^{-2}}{x!}$

~~$P(X \geq 2) = \sum_{2}$~~

$P($~~$x<t) = F_T(s) = 1 - e^{-2T}$~~

$P($~~$X \geq 2) = 1 - P(X \leq 2) = e$~~$^{-4}$

$\sigma = 20$ HRS

④ a) $\mu = 300$

$N_z\left(\frac{t-300}{20}\right)$

$P(310 < t < 330) = P\left(\frac{1}{2} < \frac{t-300}{20} < \frac{3}{2}\right)$

$= P\left(\frac{t-300}{20} < \frac{3}{2}\right) - P\left(\frac{t-300}{20} < \frac{1}{2}\right)$

$= .9332 - .6915$

$= .2417$

b) $P(t < x) = P\left(\frac{t-300}{20} < \frac{x-300}{20}\right) = .90$

$\frac{x-300}{20} = 1.28$

$x - 300 \quad 35.6$

$\Rightarrow x = 335.6$ HRS

-2

1.28
20
35.60

⑤ $P_x(x) = q^{x-1}p$

$p = \frac{1}{4},\ q = 3/4$

$p(x=1) = 3/4 \quad \frac{1}{4}$

$p(x=2) = \frac{3}{4} \cdot \frac{1}{4}$

$p(x \leq 2) = \frac{12}{16} + \frac{3}{16} = \frac{15}{16}$

$p(x > 2) = 1 - P(X \leq 2) = \frac{16}{16} - \frac{15}{16} = \frac{1}{16}$

-3

Table 3.C *Standard Normal Distribution Function (continued)*

z	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8079	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9700	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.	.9987									

1. $\mu = \sum_{x=0}^{+\infty} x \frac{e^{-\alpha}\alpha^x}{x!} = \alpha \sum_{x=0}^{+\infty} \frac{e^{-\alpha}\alpha^{x-1}}{(x-1)!}$

$= \alpha \sum_{y=0}^{+\infty} \frac{e^{-\alpha}\alpha^y}{y!} = \alpha$

$E[X(X-1)] = \sum_{x=0}^{+\infty} \frac{x(x-1)e^{-\alpha}\alpha^x}{x!}$

$= \alpha^2 \sum_{x=2}^{+\infty} \frac{e^{-\alpha}\alpha^{x-2}}{(x-2)!} = \alpha^2 \sum_{y=0}^{+\infty} \frac{e^{-\alpha}\alpha^y}{y!} = \alpha^2$

$\sigma^2 = E[X(X-1)] + E(X) - \mu^2 = \alpha^2 + \alpha - \alpha^2 = \alpha$

2. (a) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$= 1 - \binom{4}{0}(.3)^0(.7)^4 - \binom{4}{1}(.3)^1(.7)^3$

$= 1 - .2401 - .4116 = .3483$

(b) $P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$

$= \frac{\binom{3}{2}\binom{7}{2}}{\binom{10}{4}} + \frac{\binom{3}{3}\binom{7}{1}}{\binom{10}{4}} + 0 = \frac{63}{210} + \frac{7}{210} = \frac{1}{3}$

3. (a) $P(X \geq 2) = 1 - P(X=0) - P(X=1)$ $\quad \alpha = 3 \cdot \frac{2}{3} = 2$

$= 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2^1}{1!} = 1 - 3e^{-2}$

(b) $\lambda = 3$

$P(T \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} 3e^{-3t}\,dt = -e^{-3t}\Big|_0^{\frac{1}{2}}$

$= 1 - e^{-\frac{3}{2}}$

4. (a) $P(310 < X < 330) = P\left(\frac{310-300}{20} < Z < \frac{330-300}{20}\right)$

$= P\left(\frac{1}{2} < Z < \frac{3}{2}\right) = .9332 - .6915 = .2417$

(b) $P(X > k) = P\left(\frac{X-300}{20} > \frac{k-300}{20}\right) = .10$

thus $P\left(Z > \frac{k-300}{20}\right) = .10$

$P(Z > 1.282) = .10$

$1.282 = \frac{k-300}{20}$

$k = 325.64$

5. $p = \frac{13}{52} = \frac{1}{4}$

$P(Y > 2) = 1 - P(Y=1) - P(Y=2)$

$= 1 - \frac{1}{4} - \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{16}$

CHAPT. 4 PLUGS; 11-12-70

I) BINOMIAL RANDOM VARIABLE; X = NUMBER OF SUCCESSES IN n REPEATED BERNOULLI TRIALS WITH PROB. p ON EACH TRIAL OF SUCCESS

A) PARAMETERS: n, p

B) $p_X(x) = \binom{n}{x} p^x q^{n-x}$ $\quad x = 0, 1, 2, \ldots, n$

$\quad = 0$ $\quad$ OTHERWISE

C) $F_X(t) = \sum_{k \leq t} \binom{n}{k} p^k q^{n-k}$ (TABLE)

D) 1) $\mu_X = np$

2) $\sigma_X^2 = npq$

E) BERNOLLI RANDOM VARIABLE IF $n=1$

II) GEOMETRIC RANDOM VARIABLE; INDEPENDENT BERNOULLI TRIALS PERFORMED UNTIL SUCCESS

A) PARAMETER; p

B) $p_Y(k) = q^{k-1} p$ $\quad k = 1, 2, 3 \ldots$

$\quad = 0$ $\quad$ OTHERWISE

C) 1) $\mu_X = 1/p$

2) $\sigma_X^2 = q/p^2$

III) HYPERGEOMETRIC RANDOM VARIABLE: M POSSIBILITIES, W SUCCESSES IN M, n # OF EXPERIMENTS

A) PARAMETERS: M, W, n

B) $p_Z(k) = \dfrac{\binom{W}{k}\binom{M-W}{n-k}}{\binom{M}{n}}$; $k = 0, 1, 2, 3, \ldots, n$

$\left[\binom{b}{a} = 0 \text{ IFF } a > b\right]$

C) 1) $\mu_Z = \dfrac{nW}{M}$

2) $\sigma_Z^2 = n \dfrac{W}{M} \dfrac{(M-W)}{M} \dfrac{(M-n)}{(M-1)}$

IV) POISSON RANDOM VARIABLE: λ IS OBSERVED S UNITS OF TIME

A) PARAMETER: λs

B) $p_x(k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s}$ $k = 0, 1, 2, \ldots$

$= 0$ OTHERWISE

C) $F_x(t; \lambda s) = 0$ $t < 0$ (TABLE)

$= \sum_{k \leq t} p_x(t)$ $t \geq 0$

D) 1) $\mu_x = \lambda s$

2) $\sigma_x^2 = \lambda s$

V) UNIFORM RANDOM VARIABLE: X UNIFORMLY DISTRIBUTED ON $[a, b]$

A) $f_X(x) = \frac{1}{b-a}$ $a < x < b$

$= 0$ OTHERWISE

B) $F_x(x) = 0$ $x < a$

$= \frac{t-a}{b-a}$ $a \leq t \leq b$

$= 1$ $t > b$

C) 1) $\mu_x = \frac{b+a}{2}$

2) $\sigma_x^2 = \frac{(b-a)^2}{12}$

VI) EXPODENTIAL RANDOM VARIABLE:

GIVEN POISON PROCESS STARTING AT $t=0$ OBSERVING, T= TIME THAT PASSES FOR 1ST EVENT

A) PARAMETERS: λ

B) $f_T(s) = \lambda e^{-\lambda s} \mu(t)$

C) $F_T(s) = (1 - e^{-\lambda s}) \mu(t)$

D) $\mu_T = 1/\lambda$

E) $\sigma_T^2 = 1/\lambda^2$

12-7-72

THEOREM

LET X BE A BINOMIAL RANDOM VARIABLE WITH PARAMETERS n AND p, LET $Z = \frac{X - np}{\sqrt{np(1-p)}}$.
FOR LARGE n, Z IS APPROXIMIATELY A STANDARDIZED NORMAL RANDOM VARIABLE.

$$\begin{cases} n = \text{NUMBER OF TRIALS} \\ p = \text{PROBABILITY OF SUCCESS} \end{cases}$$

FOR A BINOMIAL RANDOM VARIABLE:

$$E[X] = np$$

$$VAR[X] = np(p-1)$$

$$Z_n = \frac{X_n - np}{\sqrt{np(1-p)}}$$

LET $F_{Z_n}(t)$ BE THE c.d.f. OF Z_n

THEN:

$$\lim_{n \to \infty} F_{Z_n}(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

LET X = # OF HEADS WHEN A FAIR COIN 20 TIMES

FIND: $P[8 \le X \le 11)$

NOW: $p = 0.5$ & $n = 20$

$$P[8 \le X \le 11] = P[X \le 11] - P[X \le 7]$$
$$= 0.7483 - 0.1316 \quad \text{(TABLE ON pg. 392)}$$
$$= 0.6167$$

FROM NORMAL DISTRIBUTION CONSIDERATIONS:

$$P(8 \le X_B \le 11) \approx P(8 \le X_N \le 11)$$

$$\mu = np = (20)(0.5) = 10$$

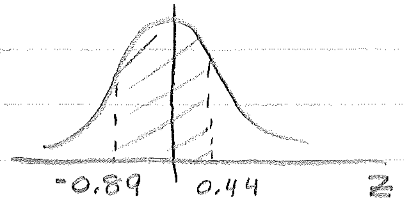
$$\sigma^2 = np(1-p) = 5 \Rightarrow \sigma = \sqrt{5} \sim 2.25 = \frac{9}{4}$$

STANDARDIZING:

$$Z = \frac{X - \bar{X}}{\sigma}$$

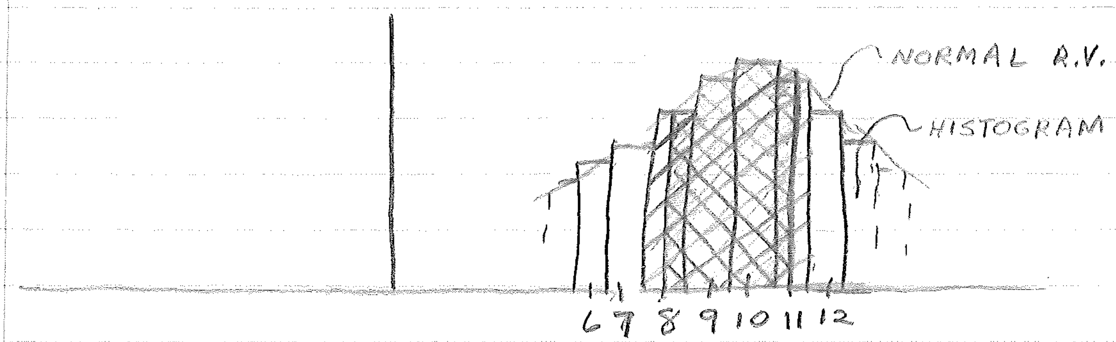
$$P(8 \le X_N \le 11) = P\left[\frac{8-10}{9/4} \le \frac{X_N - 10}{9/4} \le \frac{11-10}{9/4}\right]$$

$$= P\left[-\frac{8}{9} \le Z \le \frac{4}{9}\right] = P(-.89 \le Z \le 0.44)$$



$$P(-0.89 \le Z \le 0.44) = P(Z \le 0.44) + P(Z \le 0.89) - 1.0$$
$$= 0.6700 + 0.8133 - 1.0 \qquad (\text{Pg } 398)$$
$$= 0.4833$$

WHAT'S HAPPENING?



SHOULD USE ESTIMATION 'TWIXT $7\frac{1}{2}$ & 11.5

SOOO....

$$P(8 \le X_B \le 11) \approx P(7.5 \le X_N \le 11.5)$$
$$= P\left(\frac{7.5 - 10}{9/4} \le Z \le \frac{11.5 - 10}{9/4} \right)$$
$$= P\left(-10/9 \le Z \le 6/9 \right)$$
$$= P(1.11 \le Z \le 0.67)$$
$$= 0.2486 + 0.3665$$
$$= 0.6151$$

NOTE THE BETTER CORRELATION BETWEEN THIS AND BINOMIAL ANALYSIS, VIA USE OF THE $\frac{1}{2}$ CORRECTION FACTOR.

ASSIGNMENT:

Pp. 74-75 ; PROBS (3, 4, 6, 7, 8, 9, 10, 11, 12)

CHAPTER SIX WORDS: (6.1, 6.2)
1) FREQUENCY DISTRIBUTION
2) CLASS
3) CLASS LIMITS
4) CLASS BOUNDRIES
5) CLASS MARK
6) CLASS INTERVAL

ASSIGNMENT
PP. 114-6; PROB (3, 13a) → TO HAND IN
→ ALSO CONSTRUCT A HISTOGRAM

12-8-72

(EXAMINATION ON LAST FRIDAY BEFORE VACATION)

DATA: $x_1, x_2, \ldots, x_n$ (RAW DATA)
$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ ⇒ MEAN (CENTRAL TENDENCY)

EX: 2, 3, 1, 0, 2
$n = 5$; $\sum_{i=1}^{5} = 8$
$\Rightarrow \bar{x} = \frac{8}{5}$

MEDIAN (CENTRAL TENDENCY)
1) RANK DATA FROM LOWEST TO HIGHEST (n)
2) FOR n ODD, THE MEDIAN IS THE MIDDLE PIECE OF DATA
3) IF n IS EVEN, THE MEDIAN IS THE ARITHMETIC MEAN OF THE TWO MIDDLE PIECES OF DATA

EX:(A) 2, 3, 1, 0 2
1) 0, 1, 2, 2, 3
2) MEDIAN = 2

EX:(B) 5, 4, 9, 2, 0, 8
1) 0, 2, 4, 5, 8 9
3) MEDIAN = 4.5

MEDIUM INSENSITIVE TO EXPERIMENT MISTAKES
MEAN HAS BETTER THEORETICAL APPLICATIONS

RANGE OF DATA = BIGGEST PIECE OF DATA − SMALLEST

$\sigma^2 = \sum_{ALL\ x} (x-\mu)^2 p(x)$ (VARIANCE)

FOR ALL EQUALLY LIKELY PROBABILITIES:

$\sigma^2 = \frac{\sum (x-\mu)^2}{n}$

DEFINITION; THE VARIANCE (s^2) OF A SET OF DATA $x_1, x_2, \ldots, X_n$ IS:

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$$

S = STANDARD DEVIATION

EX)

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
2	$2/5$	$\frac{4}{25}$
3	$7/5$	$\frac{49}{25}$
1	$-3/5$	$9/25$
0	$-8/5$	$64/25$
2	$2/5$	$4/25$

NOTE $\sum_{i=1}^{n} (x_i - \bar{x}) = 0$

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{130}{25}$$

$$s^2 = \frac{1}{4} \sum_{i=1}^{n} (x_i - \bar{x})^2 = 1.3$$

$$s = \sqrt{1.3}$$

THEOREM:

$$S^2 = \frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{(n-1)}$$

PROOF: $S^2 \triangleq \frac{1}{n-1} \sum_{i=1}^{n} (x - \bar{x})^2$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i^2 - 2\bar{x}x_i + \bar{x}^2)$$

$$= \frac{1}{n-1} \left[\Sigma x_i^2 - 2\bar{x} \Sigma x_i + n\bar{x}^2 \right]$$

$$= \frac{1}{n-1} \left[\Sigma x_i^2 - 2\bar{x} n\bar{x} + n\bar{x}^2 \right]$$

$$= \frac{\Sigma x_i^2 - n\bar{x}^2}{n-1}$$

$\mu_i = \frac{x_i - A}{C}$; $\begin{pmatrix} A = 490 \\ C = 5 \end{pmatrix} \leftarrow$ CODING DATA

x_i	μ_i	μ_i^2
480	-2	4
475	-3	9
490	0	0
500	2	4
512	4	16

$\bar{\mu}_i = 1/5$

$$S_\mu^2 = \frac{\Sigma \mu_i^2 - n\bar{\mu}^2}{n-1}$$

$$= \frac{33 - 5\left(\frac{1}{5}\right)^2}{4} = \frac{41}{5}$$

$$\bar{X} = C\bar{\mu} + A = (5)\left(\tfrac{1}{5}\right) + 490 = 491$$

$$S_x^2 = C^2 S_\mu^2 = (25)\left(\tfrac{41}{5}\right) = 205$$

12-11-72

Pg. 75, #12

FIND $P[0.245 \le X \le 0.255]$

$X \Rightarrow N(0.251, (0.003)^2)$

NORMALIZING:

$$P\left[\frac{.245-.251}{0.003} \le Z \le \frac{0.255-0.251}{0.003}\right] = P[-2 \le Z \le 1.33]$$

$= 0.4082 + 0.4772$

$= 0.8854$

CENTRAL TENDENCY $\Rightarrow$ (MEDIAN, MEAN)

DISTRIBUTION $\Rightarrow$ (VARIANCE, RANGE)

GROUP DATA MEAN:

X HAS MEAN $p(x)$

$\mu = \sum_x x\, p(x)$

LET X_i = CLASS MARK OF i^{TH} CLASS

f_i = ABSOLUTE FREQUENCY OF i^{TH} CLASS

k = NUMBER OF CLASSES

$n = \sum_{i=1}^{k} f_i$ = NUMBER OF DATA

$\overline{X}_g = \sum_{i=1}^{k} f_i k / n$ = GROUP DATA MEAN

$$S_g^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \overline{X}_g)^2}{n-1}$$

THEOREM: $S_g^2 = \dfrac{\sum_{i=1}^{k} f_i x_i^2 - X_g^2}{n-1}$

IN GENERAL $\overline{X} \neq \overline{X}_g$

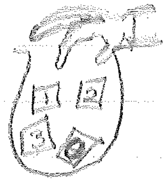
12-12-72

RANDOM SAMPLING

DEFINITION: LET X BE A RANDOM VARIABLE. THE SET OF POSSIBLE OUTCOMES OF X CAN BE THOUGHT OF AS THE SET OF POSSIBLE VALUES IN A CORRESPONDING POPULATION. THEN THE DENSITY OR PROBABILITY FUNCTION OF X IS CALLED THE DISTRIBUTION OF THE POPULATION. THE MEAN μ AND THE VARIANCE σ^2 OF X IS THE MEAN AND THE VARIANCE OF THE POPULATION. X ITSELF IS CALLED THE POPULATION RANDOM VARIABLE.

EXAMPLE

NUMBERS IN A BOWEL



x	$p(x)$
0	1/4
1	1/4
2	1/4
3	1/4

$$\mu = \sum x\, p(x) = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{4} + 2 \cdot \tfrac{1}{4} + 3 \cdot \tfrac{1}{4} = \tfrac{3}{2}$$

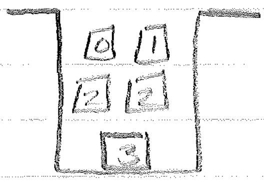
μ = POPULATION MEAN

$$\sigma^2 = E[(X-\mu)^2]$$
$$= E[X^2] - \mu^2$$

$$E[X^2] = \sum x^2 p(x) = 0^2 \tfrac{1}{4} + 1^2 \tfrac{1}{4} + 2^2 \tfrac{1}{4} + 3^2 \tfrac{1}{4} = \tfrac{14}{4}$$

$$\Rightarrow \sigma^2 = 5/4$$

$$\sigma^2 = \sum_x (x-\mu)^2 p(x)$$
$$= \frac{\sum_x (x-\mu)^2}{4}$$

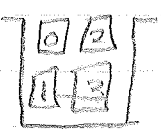


Y	$P(Y)$
0	$1/5$
1	$1/5$
2	$2/5$
3	$1/5$

RANDOM SAMPLING
w/o REPLACEMENT

DEFINITION: SUPPOSE A POPULATION HAS N DISTINCT ELEMENTS, EACH OF WHICH HAS PROBABILITY $1/N$ ASSOCIATED WITH IT. THEN A SET OF OBSERVATIONS $X_1, X_2, X_3, \dots, X_n$ $(n < N)$ CONSTITUTE A RANDOM SAMPLE OF SIZE n TAKEN WITHOUT REPLACEMENT FROM THIS FINITE POPULATION IF IT IS CHOSEN SUCH THAT EACH SUBSET OF n OF n OF THE N ELEMENTS HAS PROBABILITY $\frac{1}{\binom{N}{n}}$ OF BEING CHOSEN.

EX)



FOR: $n = 2$

(0,1)
(0,2)
(0,3)
(1,2)
(1,3)
(2,3)

SAMPLING WITH REPLACEMENT:

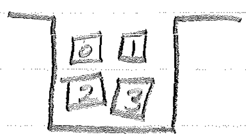
DEFINITION: LET A POPULATION HAVE DISTRIBUTION $f(x)$ (OR $p(x)$). THEN A SET OF OBSERVATIONS $x_1, x_2, \ldots, x_n$ CONSTITUTES A RANDOM SAMPLE OF SIZE n FROM AN INFINITE POPULATION, OR A FINITE POPULATION WITH REPLACEMENT IF:

(1) EACH x_i IS AN OUTCOME OF A RANDOM VARIABLE WHOSE DENSITY (OR PROBABILITY) FUNCTION IS $f(x)$ (OR $p(x)$)

(2) THESE n RANDOM VARIABLES ARE INDEPENDENT

12-14-72

EX)



$n = 2$

x_1	x_2	$p(x_1, x_2)$		
0	0	1/16	2	0	"
0	1	"	2	1	"
0	2	"	2	2	"
0	3	"	2	3	"
1	0	"	3	0	"
1	1	"	3	1	"
1	2	"	3	2	"
1	3	"	3	3	"

$$p(x_1, x_2) = P(X_1 = x_1 \ \& \ X_2 = x_2) = \frac{1}{16}$$

x_1	$p(x_1)$	x_2	$p_2(x_2)$
0	1/4	0	1/4
1	1/4	1	1/4
2	1/4	2	1/4
3	1/4	3	1/4

$p(x_1, x_2) = p_1(x_1)\, p_2(x_2)$

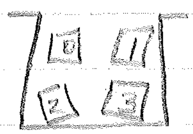
$\therefore$ X_1 AND X_2 ARE INDEPENDENT

NOTE X_1 & X_2 HAVE THE SAME PROBABILITY FUNCTION

DEFINITION: STATISTIC: A FUNCTION OF THE OBSERVATIONS IS A RANDOM SAMPLE

THEOREM 7-1: LET $\bar{x}$ BE THE MEAN OF A RANDOM SAMPLE OF n TAKEN FROM A POPULATION WITH MEAN μ AND VARIANCE σ^2. LET $\bar{X}$ BE THE SAMPLE MEAN RANDOM VARIABLE (i.e. THE RANDOM VARIABLE WHOSE OUTCOME IS $\bar{x}$). THEN $\mu_{\bar{x}} = E[\bar{X}] = \mu$. ALSO; $\sigma_{\bar{x}}^2 = \sigma^2/n$ IF THE RANDOM SAMPLING IS DONE FROM AN INFINITE POPULATION OR FROM A FINITE POPULATION WITH REPLACEMENT AND $\sigma_{\bar{x}}^2 = \frac{N-n}{N-1} \frac{\sigma^2}{n}$ IF THE RANDOM SAMPLE IS TAKEN FROM A FINITE POPULATION OF THE SIZE N WITHOUT REPLACEMENT.

(EX)



x	$p(x)$
0	1/4
1	1/4
2	1/4
3	1/4

$\mu = 3/2$

$\sigma^2 = 5/4$

SAMPLING w/o REPLACEMENT

	P(SAMPLE)	$\bar{x}$
{0, 1}	1/6	0.5
{0, 2}	1/6	1.0
{0, 3}	1/6	3/2
{1, 2}	1/6	3/2
{1, 3}	1/6	2
{2, 3}	1/6	5/2

$\bar{x}$	$p_{\bar{x}}(\bar{X} = \bar{x})$
1/2	1/6
1	1/6
3/2	1/3
2	1/6
5/2	1/6

$\mu_{\bar{x}} = \sum_{\bar{x}} \bar{x}\, p_x(\bar{x}) = 3/2$

$\mu_{\bar{x}}$ = POPULATION MEAN $= \mu$

$\sigma_{\bar{x}}^2 = E[\bar{x}^2] - \mu_{\bar{x}}^2$

$E[\bar{x}^2] = (\frac{1}{2})^2 \frac{1}{6} + 1^2 \cdot \frac{1}{6} + (\frac{3}{2})^2 \cdot \frac{2}{6} + 2^2 \frac{1}{6} + (\frac{5}{2})^2 \frac{1}{6}$

$\Rightarrow \sigma_{\bar{x}}^2 = \frac{10}{24} = 5/12$

RECALL $\quad \sigma_{\bar{x}}^2 = \frac{N-n}{N-1} \quad \frac{\sigma^2}{n}$

$= \frac{4-2}{4-1} \quad \frac{(5/4)}{2} = \frac{5}{12}$

12-15-72

SAMPLING WITH REPLACEMENT

$\boxed{\boxed{0}\ \boxed{2} \\ \boxed{1}\ \boxed{3}}$

x	$p(x)$
0	1/4
1	1/4
2	1/4
3	1/4

$\mu = \frac{3}{2}$; $\sigma^2 = 5/4$

SAMPLE	PROBABILITY		$\bar{x}$
{0, 0}	1/16		0
{0, 1}	2/16		1/2
{0, 2}	2/16	TAKING INTO ACOUNT ORDER	1
{0, 3}	2/16		3/2
{1, 1}	1/16		1
{1, 2}	2/16		3/2
{1, 3}	2/16		2
{2, 2}	1/16		2
{2, 3}	2/16		5/2
{3, 3}	1/16		3

$\bar{x}$	$P_{\bar{x}}(\bar{x})$
0	1/16
1/2	2/16
1	3/16
3/2	4/16
2	3/16
5/2	2/16
3	1/16

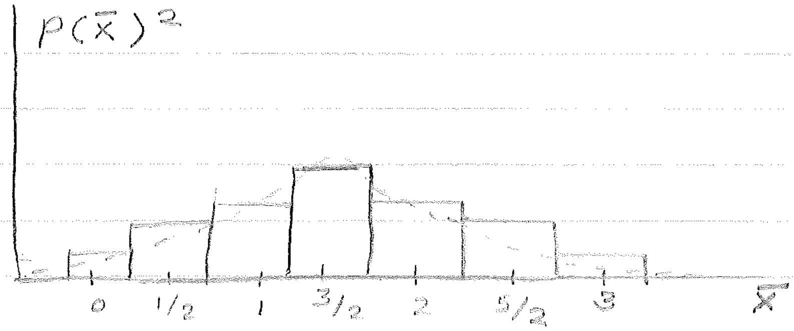
$\mu_{\bar{x}} = 3/2$

$E[\bar{x}^2] = \sum_{\bar{x}} \bar{x}^2 p_{\bar{x}}(x) = \frac{184}{64}$

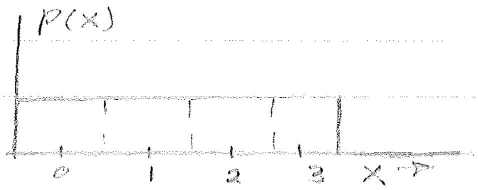
$\sigma_{\bar{x}}^2 = E[\bar{x}^2] - \mu_{\bar{x}}^2$

$= \frac{184}{64} - \frac{144}{64}$

$= \frac{40}{64} = 5/8 \qquad$ NOTE $\sigma_{\bar{x}}^2 = \sigma^2/n$



AS n INCREASES, THE HISTOGRAM APPROACHES A NORMAL CURVE.



THEOREM: (A FORM OF THE CENTRAL LIMIT THEM.)
IF $\bar{X}$ IS THE MEAN OF A RANDOM SAMPLE OF SIZE n TAKEN FROM AN INFINITE POPULATION OR WITH REPLACEMENT FROM A FINITE POPULATION WITH MEAN μ AND VARIANCE σ^2, THEN $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ IS APPROXIMATELY AN OUTCOME OF A STANDARDIZED NORMAL RANDOM VARIABLE WHEN n IS LARGE.

THEOREM: IF $\bar{X}$ IS THE MEAN OF A RANDOM SAMPLE OF SIZE n TAKEN FROM A NORMAL POPULATION WITH MEAN μ AND VARIANCE σ^2, THEN $\bar{X}$ IS EXACTLY THE OUTCOME OF A NORMAL RANDOM VARIABLE WITH MEAN μ AND VARIANCE σ^2/n (ie $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ IS AN OUTCOME OF A $N(0,1)$ RANDOM VARIABLE)

Pg 136

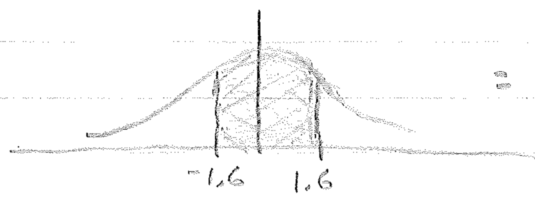
13) LET μ BE THE TRUE SPECIFIC GRAVITY OF THE METAL.

$\sigma = 0.5$; $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{.05}{\sqrt{16}} = \frac{.05}{4}$

FIND $P[|\bar{x} - \mu| < 0.02]$

$$= P\left[\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} < \frac{.02\sqrt{16}}{.05}\right]$$

$$= P\left[|z| < \frac{.02\sqrt{16}}{0.05} = \frac{8}{5} = 1.6\right]$$



$$= 2 \times 0.4452 = 0.8904$$

PROBLEMS:

Pg 135; 7, 10-14

12-18-72

s^2 SAMPLE VARIANCE

STATISTICS ARE RANDOM SAMPLES

EX) Pg 136 # 14

$n = 400$

$\bar{x}$ = MEAN OF THE SAMPLE $\Rightarrow$ WILL BE DISTRIBUTED NORMALLY

μ = POPULATION MEAN : $\bar{x} \neq \mu$ BUT $\bar{x} \sim \mu$

SO FIND $P[|\bar{x} - \mu| > 2.5] = P\left[\frac{|\bar{x}-\mu|}{\sigma/\sqrt{n}} > \frac{2.5\sqrt{n}}{\sigma}\right]$

$(\sigma_{\bar{x}} = \sigma/\sqrt{n})$

$$= P\left[|z| > \frac{2.5\sqrt{400}}{18}\right]$$

$$= P[|z| > 2.78]$$

$$= 2(.0027)$$

$$= 0.0054$$

DISTRIBUTIONS:

<u>DEFINITION:</u> A CHI-SQUARED RANDOM VARIABLE WITH PARAMATER γ IS A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION OF THE FORM:

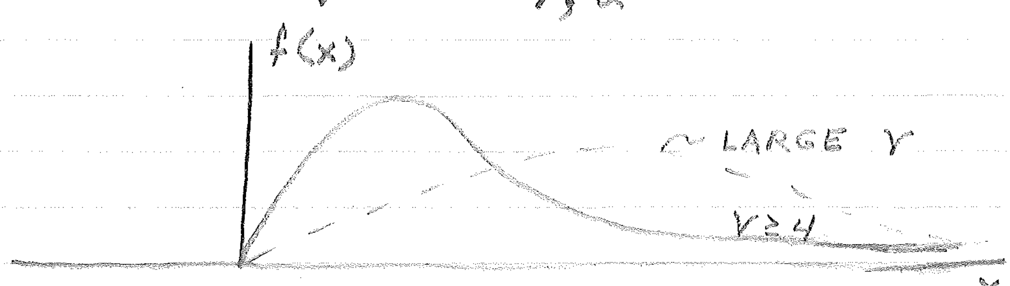
$$f(x) = \begin{cases} \frac{1}{2^{\gamma/2}\,\Gamma(\frac{\gamma}{2})} x^{\frac{\gamma}{2}-1} e^{-x/2} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\left\{ \begin{array}{l} \Gamma(\alpha) = \int_0^\infty Y^{\alpha-1} e^{-Y} dY \\ \text{IF } \alpha \in \text{POSITIVE INTEGER: } \Gamma(\alpha) = (\alpha-1)! \end{array} \right\}$$

<u>REMARKS</u>

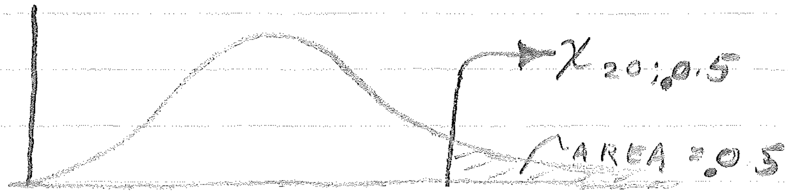
1) IN GENERAL, $\gamma \in$ POSITIVE INTEGER
2) WE USE THE NOTATION χ^2_γ TO REPRESENT A CHI-SQUARED RANDOM VARIABLE WITH PARAMETER γ
3) THE NOTATION $\chi_{\gamma;\alpha}$ REPRESENTS THE # $\ni$ THE:

$$P[\chi^2_\gamma > \chi^2_{\gamma;\alpha}] = \alpha$$



4) FOR γ LARGE, DISTRIBUTION, X_γ APPROACHES NORMAL DISTRIBUTION

EX) $\nu = 20$



$\chi_{20;0.5} = 31.40$ FROM PG 400

THEOREM: THE SQUARE OF A STANDARDIZED NORMAL RANDOM VARIABLE IS A χ_1^2 RANDOM VARIABLE

THEOREM: LET $Z_1, Z_2, \ldots, Z_\nu$ BE ν INDEPENDENT STANDARDIZED NORMAL RANDOM VARIABLES. THEN $Z_1^2 + Z_2^2 + \ldots + Z_\nu^2$ IS A χ_ν^2 RANDOM VARIABLE.

(INDEPENDANCE $P[X, Y] = P[X]P[Y]$

ν = DEGREES OF FREEDOM

THEOREM 7.4: IF S^2 IS THE VARIANCE OF A RANDOM SAMPLE OF SIZE n TAKEN FROM A NORMAL POPULATION HAVING VARIANCE σ^2, THEN $\frac{(n-1)S^2}{\sigma^2}$ IS AN OUTCOME OF A χ_{n-1}^2 RANDOM VARIABL

12-19-72

TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION; $X_1, X_2, \ldots, X_n$ ARE THE RANDOM SAMPLE OBSERVATIONS

CONSIDER:

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$$

EACH X_i IS A $N(\mu, \sigma^2)$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i^2$$

$$= \chi_n^2$$

CONSIDER: $$\frac{(n-1)S^2}{\sigma^2} = \frac{(n-1)\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{(n-1)}}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2$$

$$= \chi_{n-1}^2$$

(μ = POPULATION MEAN; $\bar{X}$ = SAMPLE MEAN)

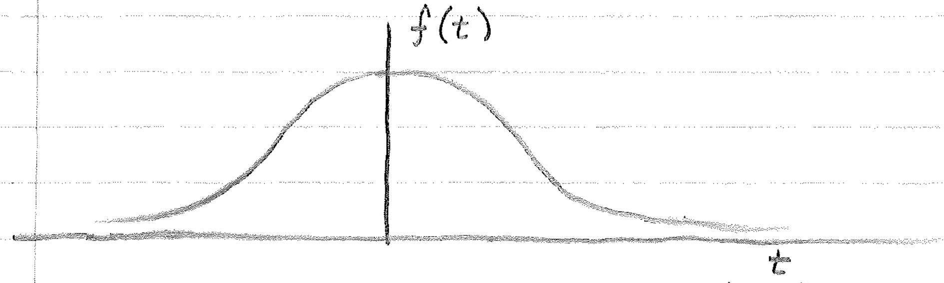
<u>DEFINITION</u>: LET Z BE A STANDARDIZED NORMAL RANDOM VARIABLE AND LET χ_γ^2 BE AN INDEPENDENT CHI-SQUARED RANDOM VARIABLE WITH γ DEGREES OF FREEDOM (df)

THEN: $$T_\gamma = \frac{Z}{\sqrt{\chi_\gamma^2/\gamma}}$$

IS CALLED A STUDENT t RANDOM VARIABLE WITH γ d.f.

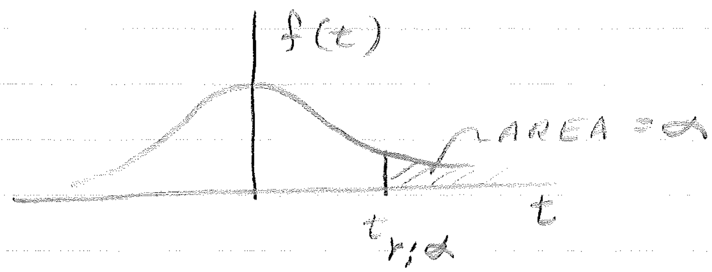
<u>THEOREM</u>: $\Longrightarrow$ THE DENSITY FUNCTION OF A T_γ RANDOM VARIABLE IS:

$$f(t) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\sqrt{\pi\gamma}}\left[1 + \frac{t^2}{\gamma}\right]^{-\left(\frac{\gamma+1}{2}\right)} \quad ; -\infty < t < \infty$$



$$\lim_{r \to \infty} f_r(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

REMARK: WE WILL USE THE NOTATION $t_{r;\alpha}$ TO REPRESENT THE REAL NUMBER ∋ THE $P[T_r > t_{r;\alpha}] = \alpha$



$$t_{r;0.95} = -t_{r;0.05}$$

THEOREM (7.3): IF $\bar{X}$ IS THE MEAN AND S^2 IS THE VARIANCE OF A RANDOM SAMPLE OF n TAKEN FROM A $N(\mu, \sigma^2)$ POPULATION, THEN $t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ IS AN OUTCOME OF A STUDENT'S t RANDOM VARIABLE WITH $r = n-1$ DEGREES OF FREEDOM.

NOTE $\lim_{n \to \infty} \frac{\bar{X} - \mu}{S/\sqrt{n}} = \lim_{n \to \infty} \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$

(OVER FOR PROOF)

PROOF: $$\frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right]}{\sqrt{\frac{(n-1)s^2}{\sigma^2}\Big/(n-1)}}$$

$$= \frac{z}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}}$$

$$= t_{n-1}$$

12-21-72

EXAM TOMORROW - BRING SLIDE RULES

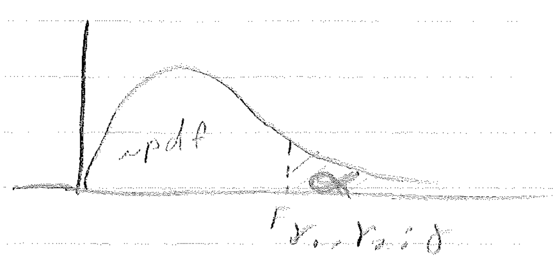
THE F DISTRIBUTION

DEFINITION: LET $\chi^2_{\gamma_1}$ & $\chi^2_{\gamma_2}$ BE INDEPENDENT CHI-SQUARE RANDOM VARIABLES WITH γ_1 & γ_2 d.f., RESPECTIVELY.

THEN;

$$F = \frac{\left[\chi^2_{\gamma_1}/\gamma_1\right]}{\left[\chi^2_{\gamma_2}/\gamma_2\right]}$$

IS AN F RANDOM VARIABLE WITH γ_1 & γ_2 d.f.



REMARK: WE USE THE NOTATION F_{γ_1,γ_2}.

$F_{\gamma_1,\gamma_2;\alpha}$ IS THE NUMBER $\ni P\left[F_{\gamma_1,\gamma_2} > F_{\gamma_1,\gamma_2;\alpha}\right] = \alpha$

THEOREM: $F_{\nu_1,\nu_2;\alpha} = \frac{1}{F_{\nu_2,\nu_1;1-\alpha}}$

THEOREM 2.5: IF S_1^2 & S_2^2 ARE THE VARIANCES OF RANDOM SAMPLES OF n_1 & n_2, RESPECTIVLLY, FROM INDEPENDENT NORMAL POPULATION WITH THE SAME VARIANCE, THEN $F = \frac{S_1^2}{S_2^2}$ IS AN F_{n_1-1,n_2-1} RANDOM VARIABLE.

PROOF:

$$\frac{S_1^2}{S_2^2} = \frac{\frac{(n_1-1)S_1^2}{\sigma^2} \Big/ (n_1-1)}{\frac{(n_2-1)S_2^2}{\sigma^2} \Big/ (n_2-1)}$$

$$= \frac{\chi^2_{n_1-1} / (n_1-1)}{\chi^2_{n_2-1} / (n_2-1)}$$

$$= F_{n_1-1,n_2-1}$$

9-8-73

$\bar{x}$ IS A POINT ESTIMATE OF μ, SINCE $E[\bar{x}] = \mu$

DEFINITION: LET $\hat{\theta}$ BE A POINT ESTIMATE OF A POPULATION PARAMETER θ. THEN $\hat{\theta}$ IS SAID TO BE AN UNBIASED ESTIMATE OF θ IF $E[\hat{\theta}] = \theta$. IF $E[\hat{\theta}] \neq \theta$, THEN $\hat{\theta}$ IS BIASED.

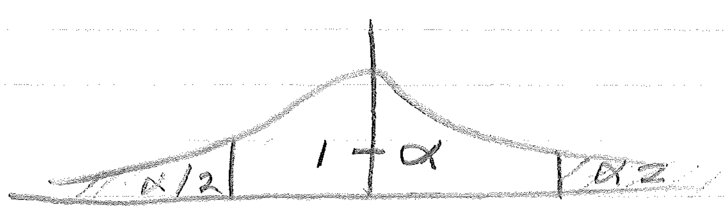
THEM: IF WE TAKE A RANDOM SAMPLE WITH REPLACEMENT FROM A FINITE POPULATION, OR A RANDOM SAMPLE FROM AN INFINITE POPULATION; THE $E[s^2] = \sigma^2$

HOWEVER, $E[s] \neq \sigma$

DEFINITION: A CONFIDENCE INTERVAL FOR A POPULATION PARAMETER IS A RANDOM INTERVAL WHICH COVERS THE TRUE VALUE OF THE PARAMETER WITH A SPECIFIED PROBIBILITY $1-\alpha$.

SUPPOSE WE ARE SAMPLING FROM A POPULATION WITH UNKNOWN MEAN μ AND KNOWN VARIANCE σ^2.

$$P(-z_{\alpha/2} \leq z \leq z_{\alpha/2}) = 1-\alpha$$



THUS $P\left(-z_{\alpha/2} \le \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right) = 1-\alpha$

$$P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{x}-\mu \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$P\left(-\bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n} \le -\mu \le -\bar{x} + z_{\alpha/2}\,\sigma/\sqrt{n}\right)$$

$$= P\left[\bar{x} + z_{\alpha/2}\,\sigma/\sqrt{n} \ge \mu \ge \bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n}\right]$$

$$\Rightarrow P(\bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n}) \le \mu \le \bar{x} + z_{\alpha/2}\,\sigma/\sqrt{n})) = 1-\alpha$$

1-9-73

1-11-73

WE KNOW σ^2

$$P\left[\bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\frac{\alpha}{2}}\,\sigma/\sqrt{n}\right) = 1-\alpha$$

CONFIDENCE INTERVAL EXPRESSED AS:

$$\bar{x} \pm z_{\alpha/2}\,\sigma/\sqrt{n}$$

WE DO NOT KNOW σ^2 FROM A NORMAL POPULATION

$$P\left[-t_{n-1;\alpha/2} \le T_{n-1} \le t_{n-1;\alpha/2}\right] = 1-\alpha$$

$$P\left[-t_{n-1;\alpha/2} \le \frac{\bar{x}-\mu}{s/\sqrt{n}} \le t_{n-1;\alpha/2}\right] = 1-\alpha$$

$$P\left[\bar{x} - t_{n-1;\frac{\alpha}{2}}\,s/\sqrt{n} \le \mu \le \bar{x} + t_{n-1;\frac{\alpha}{2}}\,s/\sqrt{n}\right] = 1-\alpha$$

Pg 149-50: 2, 3, 4, 5, 7, 11, 12, (8, 9, 10)

TEST OF HYPOTHESIS

EXAMPLE (FOR MOTIVATION?)

400 HEIGHTS OF GIRLS @ ST. MARY'S

UNIQUE μ & σ^2

TAKE A SAMPLE OF 16 GIRLS

COMPUTE $\bar{X}$ & S^2

$\begin{cases} \text{PRESIDENT SAYS } \mu = 63'' \text{; } H_0 \\ \text{REGISTRAR SAYS } \mu > 63'' \text{; } H_1 \end{cases}$

WHICH IS CORRECT? MUST EXECUTE HYPOTHESIS

H_0 IS THE NULL HYPOTHESIS ; $\mu = 63$

H_1 " " ALTERNATIVE HYPOTHESIS ; $\mu > 63$

DECISION RULE: REJECT H_0 IS $\bar{X} > 64.645$. OTHERWISE, ACCEPT H_0.

ALTERNATIVES

		STATE OF NATURE	
		H_0 IS TRUE	H_0 IS FALSE
DECISION	H_0 TRUE	NO ERROR	TYPE 2 ERROR
	H_0 FALSE	TYPE 1 ERROR	NO ERROR

TYPE 1 ERROR- REJECTING NULL HYPOTHISIS WHEN IT IS TRUE

TYPE 2 ERROR- ACCEPTING NULL HYPOTHESIS WHEN IT IS FALSE

WE USE THE NOTATION:

$$\alpha = P[\text{TYPE 1 ERROR}]$$
$$\beta = P[\;"\;\; 2 \;\;"\;]$$

BACK TO PROBLEM:

$$\alpha = P[\text{TYPE 1 ERROR}]$$
$$= P[\bar{X} > 64.645 \text{ WHEN } \mu = 63]$$

ASSUME $\sigma = 4 \Rightarrow \sigma_{\bar{x}} = \sigma/\sqrt{n} = 1$

$$\therefore \alpha = \left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = Z > \frac{64.645-63}{1} = 1.645\right] = 0.05$$

1-11-73

NULL HYPOTHESIS $\Rightarrow$

$$\left.\begin{array}{l} H_0 ; \mu = \mu_0 \\ H_1 ; \mu > \mu_0 \end{array}\right\}$$
$$\left.\begin{array}{l} H_0 ; \mu = \mu_0 \\ H_1 ; \mu < \mu_0 \end{array}\right\} \rightarrow \text{TEST OF HYPOTHESIS}$$
$$\left.\begin{array}{l} H_0 ; \mu = \mu_0 \\ H_1 ; \mu \neq \mu_0 \end{array}\right\}$$

OPERATING CHARACTERISTIC CURVE FOR THE TEST $H_0 ; \mu = \mu_0$ IS THE GRAPH OF $g(\mu) = P(\text{ACCEPTING } H_0)$

EXAMPLE:

H_0; $\mu = 63''$

H_1; $\mu > 63''$; $\sigma = 4$; $n = 16$

REJECT H_0 WHEN $\bar{x} > 64.645$

$\alpha = P[\text{REJECTING } H_0 \text{ WHEN } H_0 \text{ IS TRUE}]$

$= P[\bar{x} > 64.645 \text{ WHEN } \mu = 63''] = 0.05$

μ $g(\mu)$: $P[\text{ACCEPTING } H_0 \text{ AS A FUNCTION OF } \mu]$

63   0.95

64   $g(64) = P[\text{ACCEPTING } H_0 \text{ WHEN } \mu = 64]$

$= P[\bar{x} < 64.5 \text{ GIVEN } \mu = 64]$

$= P\left[\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{64.645 - 64}{1}\right]$

$= P[Z < 0.645]$

$\simeq 0.74$

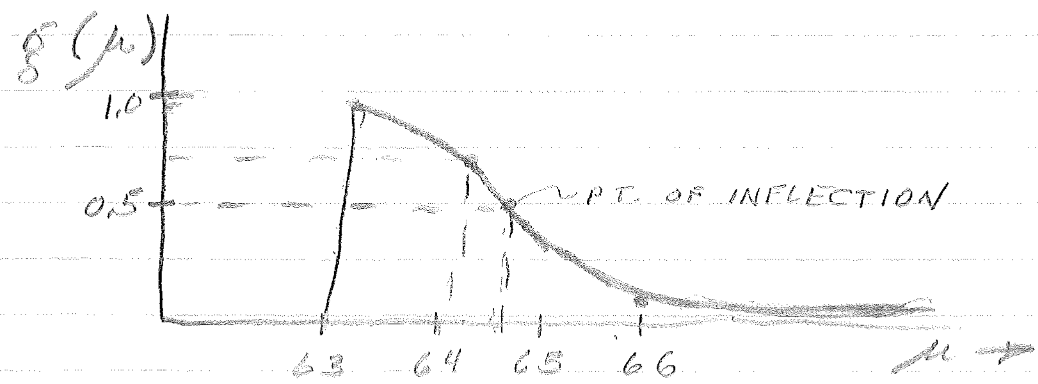
64.645   0.50

66   $g(66) = P[\text{ACCEPTING } H_0 \text{ WHEN } \mu = 66]$

$= P\left[Z < \frac{64.645 - 66}{1}\right]$

$= P[Z < 1.355]$

$\simeq 0.09$



POWER = $1 - \beta$

EXAMPLE:

$H_0: \mu = 63''$

$H_1: \mu \neq 63''$

$n = 16;\ \sigma = 4$

REJECT H_0 IF $\bar{X} > 64.96$ OR IF $\bar{X} < 61.04$

$\alpha = P[\text{TYPE 1 ERROR}]$

$= P[\text{REJECTING } H_0 \text{ WHEN IT IS TRUE}]$

$= P[\bar{X} > 64.96'' \text{ OR } \bar{X} < 61.04'' \text{ GIVEN } \mu = 63'']$

$= P[\bar{X} > 64.96''] + P[\bar{X} < 61.04]$

$= P\left[Z > \frac{64.96 - 63}{1}\right] + P\left[Z < \frac{61.04 - 63}{1}\right]$

$= P[Z > 1.96] + P[Z < -1.96]$

$= 0.025 + 0.025$

$= 0.05$

LET $\mu = 63.5$

$\beta(63.5) = P[\text{ACCEPTING } H_0 \text{ WHEN } \mu = 63.5]$

$= P[61.04 < \bar{X} < 64.96]$

$= P[-2.46 < Z < 1.46]$

$= 0.9279 - 0.0069$

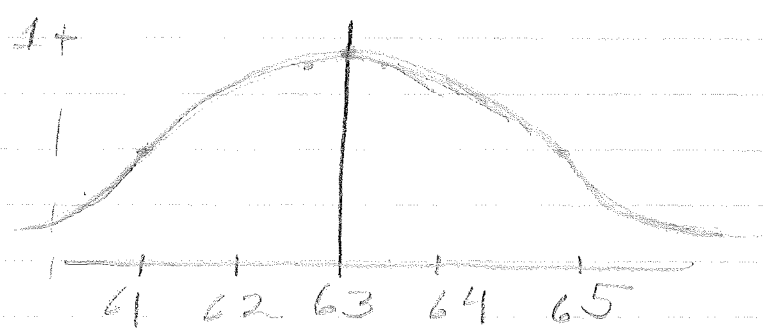
$= 0.921$

$\beta(64.96) = P[61.04 < \bar{X} < 64.96 \text{ WHEN } \mu = 64.96]$

$= P[-3.96 < Z < 0]$

≈ 0.50

μ	$\beta(\mu)$
63	0.95
63.5	0.92
64.96	0.5
62.5	0.92
61.04	0.5



(SEC. 8.2)

Pg 160-1: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11 (TO HAND IN)
DUE THURSDAY

1-15-73

TEST OF HYPOTHESIS

$H_0: \mu = \mu_0$; $H_1: \mu > \mu_0$

ASSUME σ IS KNOWN

REJECT H_0 WHEN $\bar{X} > K$, OR REJECT IF $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{K-\mu}{\sigma/\sqrt{n}}$.

WE WANT $k' = \frac{K-\mu_0}{\sigma/\sqrt{n}} \ni P\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > k' \text{ WHEN } \mu = \mu_0\right] = \alpha$

$\Rightarrow k' = Z_\alpha$

SUPPOSE WE WANT TO TEST $H_0: \mu = \mu_0$ AND σ IS KNOWN, AS WELL AS n. SET α. COMPUTE $Z = (\bar{X} - \mu_0)/\sigma/\sqrt{n}$.

IF $H_1: \mu > \mu_0$, REJECT H_0 IF $Z > Z_\alpha$.
IF $H_1: \mu < \mu_0$, REJECT H_0 IF $Z < Z_\alpha$.
IF $H_1: \mu \neq \mu_0$, REJECT H_0 IF $Z > Z_{\alpha/2}$ OR $Z < -Z_{\alpha/2}$

(SEE Pg 164)

SUPPOSE WE TAKE A RANDOM SAMPLE OF n (n IS SMALL; $n \leq 30$) FROM NORMAL POPULATION WITH UNKNOWN MEAN μ AND VARIANCE σ^2. WE WANNA TEST $H_0: \mu = \mu_0$. SET α.

COMPUTE $t = \frac{\bar{X}-\mu}{s/\sqrt{n}}$.

IF $H_1: \mu > \mu_0$, REJECT H_0 IF $t > t_{n-1;\alpha}$
IF $H_1: \mu > \mu_0$, REJECT H_0 IF $t < -t_{n-1;\alpha}$
IF $H_1: \mu \neq \mu_0$, REJECT H_0 IF $t > t_{n-1;\frac{\alpha}{2}}$ OR $t > -t_{n-1;\frac{\alpha}{2}}$

EX: $H_0: \mu = 20$; $H_1: \mu \neq 20$ (ON Pg 164)

$n = 25$. SET $\alpha = 0.01$

EXAM NEXT THURS.

1-13-73

SEC. 8-5

THEM: LET X_1 AND X_2 BE INDEPENDENT RANDOM VARIABLES WITH MEANS μ_1 & μ_2 AND VARIANCES σ_1^2 & σ_2^2. THEN $X_1 \pm X_2$ HAS MEAN $\mu_1 \pm \mu_2$ AND VARIANCE $\sigma_1^2 + \sigma_2^2$
LET X_1 & X_2 BE INDEPENDENT $N(\mu_1, \sigma_1^2)$ & $N(\mu_2, \sigma_2^2)$, THEN $X_1 \pm X_2$ IS $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

COROLLARY: IF WE TAKE RANDOM SAMPLES OF n_1 & n_2 FROM INDEPENDENT $N(\mu_1, \sigma_1^2)$ & $N(\mu_2, \sigma_2^2)$ POPULATIONS, THEN $\bar{X}_1 - \bar{X}_2$ IS A $N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$ RANDOM VARIABLE.

SUPPOSE WE TAKE RANDOM SAMPLES OF n_1 & n_2 FROM INDEPENDENT POPULATIONS 1 & 2, WHOSE VARIANCES σ_1^2 & σ_2^2, ARE KNOWN. WE WISH TO TEST:

$H_0 : \mu_1 - \mu_2 = \delta \ni \delta$ IS A GIVEN NUMBER.

WE PROCEED AS FOLLOWS: SET α, AND THEN COMPUTE $Z = \frac{\bar{x}_1 - \bar{x}_2 - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$.

$$\begin{cases} \text{IF } H_1 : \mu_1 - \mu_2 > \delta, \text{ REJECT } H_0 \text{ IF } Z > Z_\alpha \\ \text{IF } H_1 : \mu_1 - \mu_2 < \delta, \text{ REJECT } H_0 \text{ IF } Z < -Z_\alpha \\ \text{IF } H_1 : \mu_1 - \mu_2 \neq \delta, \text{ REJECT } H_0 \text{ IF } Z > Z_{\alpha/2} \text{ OR } Z < -Z_{\alpha/2} \end{cases}$$

REMARK: WE CAN USE THIS TEST IF WE HAVE NEAR NORMAL POPULATIONS, AND $n_1 > 30$ & $n_2 > 0$, EVEN IF WE DON'T KNOW σ_1^2 & σ_2^2 BY SUBSTITUTION OF S_1^2 FOR σ_1^2 AND S_2^2 FOR σ_2^2

1-17-73

~Pg 166

Ex) $H_0: \mu_1 - \mu_2 = 0$

$H_1: \mu_1 - \mu_2 \neq 0$

SET $\alpha = 0.05$

$Z_{\alpha/2} = 1.92$

$\Rightarrow$ REJECT IF $Z > 1.98$ OR $Z < -1.96$

$$Z = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n}}} \cong \frac{\bar{X}_1 - X_2}{\sqrt{\frac{S_1^2}{n_1^2} + \frac{S_2^2}{n_2}}} = 1.69$$

REJECT OR ACCEPT?

THEOREM: IF WE TAKE RANDOM SAMPLES n_1 & n_2 FROM INDEPENDENT NORMAL POPULATIONS $N(\mu_1, \sigma_1^2)$ AND $N(\mu_2, \sigma_2^2)$ WHERE $\sigma_1^2 = \sigma_2^2$, THEN:

$$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S\sqrt{1/n_1 + 1/n_2}}$$

WHERE:

$$S^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

IS AN OUTCOME OF A $T_{n_1 + n_2 - 2}$ RANDOM VARIABLE

NOTE: $(n_1 - 1)S_2^2 = \Sigma(X_{2i} - \bar{X}_2)^2$

PROOF:

$$\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{\left[\frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}\right]}{S/\sigma}$$

$$= \frac{Z}{S/\sigma}$$

$$\frac{S^2}{\sigma^2} = \frac{\frac{(n_1 - 1)S_1^2}{\sigma^2} + \frac{(n_2 - 1)S_2^2}{\sigma^2}}{n_1 + n_2 - 2}$$

$$= \frac{\chi^2_{n_1 - 1} + \chi^2_{n_2 - 1}}{n_1 + n_2 - 2}$$

$$= \frac{\chi^2_{n_1 + n_2 - 2}}{n_1 + n_2 - 2}$$

$$\therefore \quad \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S\sqrt{1/n_1 + 1/n_2}} = \frac{Z}{\sqrt{\frac{\chi^2_{n_1 + n_2 - 2}}{n_1 + n_2 - 2}}}$$

$$= T_{n_1 + n_2 - 2}$$

(S.W.: W.C.)

SUPPOSE WE TAKE RANDOM SAMPLES OF n_1 AND n_2 FROM INDEPENDENT $N(\mu_1, \sigma_1^2)$ AND $N(\mu_2, \sigma_2^2)$ POPULATIONS WHERE $\sigma_1^2 = \sigma_2^2$, WHICH ARE UNKNOWN. WE WANT TO TEST $H_0: \mu_1 - \mu_2 = \delta$. IN ORDER TO DO THIS, FIRST SET α. COMPUTE:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - \delta}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\begin{cases} \text{IF } H_1: \mu_1 - \mu_2 > \delta, \text{ REJECT } H_0 \text{ IF } t > t_{n_1+n_2-2;\alpha} \\ \text{IF } H_1: \mu_1 - \mu_2 < \delta, \text{ REJECT } H_0 \text{ IF } t < t_{n_1+n_2-2;\alpha} \\ \text{IF } H_1: \mu_1 - \mu_2 \neq \delta, \text{ REJECT } H_0 \text{ IF } t > t_{n_1+n_2-2;\frac{\alpha}{2}} \\ \qquad \text{OR IF } t < -t_{n_1+n_2-2;\frac{\alpha}{2}} \end{cases}$$

PROB. 1-8, 11-17 PG 172

1-18-73

TEST NEXT THURSDAY (OPEN BOOK - BRING SLIDERULES)

NO CLASS ON FRIDAY

TEST ON CHAPT. 8 & 9

FOR $X_1: N(\mu_1, \sigma_1^2)$ $\quad X_2: N(\mu_2, \sigma_2^2)$

AND X_1 AND X_2 ARE INDEPENDENT, THEN

$X_1 - X_2$ IS $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

FOR NON-INDEPENDENCE:

$X_1 - X_2$ IS $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2 - 2\,\mathrm{cov}(x_1, x_2))$

$\ni \mathrm{cov}(XY) = E[(\bar{X} - \mu_x)(\bar{Y} - \mu_y)]$

$= E[\bar{X}\bar{Y}] - \mu_x \mu_y$

$\rho(x, y) = \mathrm{cov}(x, y) / \sigma_x \sigma_y \ni \rho$ = CORRELATION

X_{1i} , Y_{1i}, $\Rightarrow d_i = X_{1i} - X_{2i} \Rightarrow E(d_i) = \mu_1 - \mu_2$

$H_0 : \mu_1 - \mu_2 = \delta$

$t = \dfrac{\bar{d} - \delta}{S_d/\sqrt{n}}$ ∋ n = NUMBERED PAIRS.

Pg 173

18) $H_0 : \mu_B - \mu_A = 0$

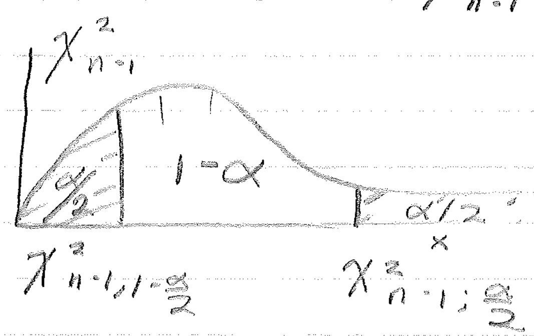
$H_1 : \mu_B - \mu_A > 0$

HAND IN PROB. 20, Pg 174

ESTIMATION OF VARIANCES

SUPPOSE WE TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION WITH μ & σ^2 UNKNOWN.

$\dfrac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} RANDOM VARIABLE



$$P[\chi^2_{n-1;1-\frac{\alpha}{2}} \le \chi^2_{n-1} \le \chi^2_{n-1;\frac{\alpha}{2}}] = 1-\alpha$$

$$P[\chi^2_{n-1;1-\alpha/2} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{n-1;\frac{\alpha}{2}}] = 1-\alpha$$

[CONSIDER $a, b, c > 0$
$a/b > c/d \Rightarrow b/a < d/c$]

THUS:

$$P\left[\frac{1}{\chi^2_{n-1;1-\alpha/2}} \geq \frac{\sigma^2}{(n-1)S^2} \geq \frac{1}{\chi^2_{n-1;\alpha/2}}\right] = 1-\alpha$$

$$= P\left[\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\right] = 1-\alpha$$

1-22-73

HYPOTHESIS CONCERNING ONE VARIABLE

SUPPOSE WE WANT TO TEST
$H_0; \sigma^2 = \sigma_0^2$ WHEN WE TAKE A R.S.
OF N FROM A $N(\mu, \sigma^2)$ POPULATION
AND COMPUTE $\chi^2 = \frac{(n-1)S^2}{\sigma_0^2}$

IF $H_1: \sigma^2 > \sigma_0^2$; REJECT H_0 IF $\chi^2 > \chi^2_{n-1;\alpha}$
IF $H_1: \sigma^2 < \sigma_0^2$; REJECT H_0 IF $\chi^2 < \chi^2_{n-1;1-\alpha}$
IF $H_1: \sigma^2 \neq \sigma_0^2$, REJECT H_0 IF $\chi^2 > \chi^2_{n-1;\alpha/2}$
OR IF $\chi^2 < \chi^2_{n-1;1-\alpha/2}$

IF $n > 31$ AND POP $\simeq N(\mu, \sigma^2)$
SET σ AND COMPUTE $Z = \frac{S - \sigma}{\sigma_0/\sqrt{2n}}$

HYPOTHESIS CONCERNING TWO VARIANCES
SUPPOSE WE TAKE A R.S. OF n_1 & n_2 FROM
INDEPENDENT $N(\mu_1, \sigma_1^2)$ AND $N(\mu_2, \sigma_2^2)$ POPULATIONS
TO TEST THE HYPOTHESIS $H_0; \sigma_1^2 = \sigma_2^2$
IN ORDER TO DO THIS, COMPUTE $F = S_1^2/S_2^2$

IF $H_1: \sigma_1^2 > \sigma_2^2$, REJECT H_0 IF $F > F_{n_1-1;n_2-2;\alpha}$
IF $H_1: \sigma_1^2 < \sigma_2^2$, REJECT H_0 IF $F < F_{n_1-1;n_2-2;1-\alpha}$
IF $H_1; \sigma_1^2 \neq \sigma_2^2$, REJECT H_0 IF $F < F_{n_1-1,n_2-1,1-\alpha/2}$
OR IF $F > F_{n_1-1;n_2-2;\alpha/2}$

NOTE: $F_{n_1-1,n_2-1,1-\alpha} = \frac{1}{F_{n_2-1,n_1-1;\alpha}}$

1-23-73

Pg. 184

8) $n_a = 10$ $n_b = 8$

$s_a^2 = 16$ $s_b^2 = 25$

$H_0 : \sigma_A^2 = \sigma_B^2$

$H_1 : \sigma_B^2 \neq \sigma_B^2$; $\alpha = 0.2$

$$F = \frac{S_B^2}{S_A^2} = \frac{25}{16} = 1.57$$

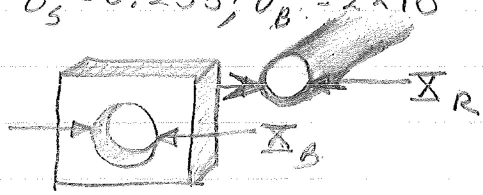
$$F_{n_B - 1; n_A - 1; \alpha/2} = F_{7,9;0.01} = 5.61$$

$\therefore$ ACCEPT H_0

Pg. 172

11) $\mu_S = 0.249$; $\mu_B = 3 \times 10^{-3}$

$\sigma_S = 0.255$; $\sigma_B = 2 \times 10^{-3}$



FIND $P[X_R > X_B] = P[X_R - X_B > 0]$

$$X_S - X_B \rightarrow N[\mu_S - \mu_B, \sigma_S^2 + \sigma_B^2]$$

$$P[X_R - X_B = Y > 0] = P\left[Z > \frac{0 - \mu_Y}{\sigma_Y}\right]$$

$$= P\left[Z > \frac{-(\mu_S - \mu_B)}{\sqrt{\sigma_S^2 + \sigma_B^2}}\right]$$

$$= P\left[Z > 6/\sqrt{13}\right]$$

$$= P[Z > 1.67]$$

$$= 0.0475$$

TEST ON CHAPTER'S 8 & 9

CHAPTER 10

ESTIMATE P: TRUE PROBABILITY OF SUCCESS IN A BINOMIAL EXPERIMENT OF n INDEPENDENT TRIALS

LET X = NUMBER OF SUCCESSES IN THE n TRIALS

$E[X] = np$

THUS $p = E\left[\frac{X}{n}\right]$

SUPPOSE n IS LARGE, THEN $\frac{X-np}{\sqrt{p(1-p)n}} = Z$ IS APPROXIMATELY A $N(0,1)$

$$P[-Z_{\alpha/2} \le Z \le Z_{\alpha/2}] = 1-\alpha$$

$$= P\left[-Z_{\alpha/2} \le \frac{X-np}{np(1-p)} \le Z_{\alpha/2}\right] \approx 1-\alpha$$

1-29-73

$$Z = \frac{x-np}{\sqrt{np(1-p)}} \approx N(0,1) \quad \text{FOR } n \gg 1$$

$$P[-Z_{\alpha/2} \le Z \le Z_{\alpha/2}] = 1-\alpha$$

$$\therefore 1-\alpha \approx P\left[-Z_{\alpha/2} \le \frac{x-np}{\sqrt{np(1-p)}} \le Z_{\alpha/2}\right]$$

SOLVING FOR p

$$1-\alpha \approx P\left[\frac{x + \frac{1}{2}Z^2_{\alpha/2} - Z_{\alpha/2}\sqrt{\frac{x(n-x)}{n} + \frac{1}{4}Z^2_{\alpha/2}}}{n + Z^2_{\alpha/2}} \le p \le \frac{x + \frac{1}{2}Z^2_{\alpha/2} + Z_{\alpha/2}\sqrt{\frac{x(n-x)}{n} + \frac{1}{4}Z^2_{\alpha/2}}}{n + Z^2_{\alpha/2}}\right]$$

TOO HAIRY!

CONSIDER:

$$\frac{x + \frac{1}{2} z^2_{1/2\alpha} \pm z_{\alpha/2}\sqrt{\frac{x(n-x)}{n} + \frac{1}{4} z^2_{\alpha/2}}}{n + z^2_{\alpha/2}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \frac{\frac{x}{n} + \frac{\frac{1}{2} z^2_{\alpha/2}}{n} \pm z_{\alpha/2}\sqrt{\frac{\left(\frac{x}{n}\right)\left(1-\frac{x}{n}\right)}{n} + \frac{\frac{1}{4} z^2_{\alpha/2}}{n^2}}}{1 + z^2_{\alpha/2}/n}$$

FOR N VERY LARGE: $\frac{\frac{1}{2} z^2_{\alpha/2}}{n} \simeq \frac{\frac{1}{4} z^2_{\alpha/2}}{n^2}$

SO ARUMENT ABOVE BECOMES:

$$\frac{x}{n} \pm z_{\alpha/2}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n}}$$

; LET $\hat{p} = \frac{x}{n} = E[p$

SO CONFIDENCE INTERVAL FOR n LARGE:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

ALTERNATE DERIVATION:

AGAIN:

$$P\left[-z_{\alpha/2} \le \frac{x - np}{\sqrt{np(1-p)}} \le z_{\alpha/2}\right] \simeq 1-\alpha$$

$$= P\left[-z_{\alpha/2} \le \frac{\frac{x}{n} - p}{\sqrt{\frac{p(1-p)}{n}}} \le z_{\alpha/2}\right]$$

NOTE:

$$p = .5 \Rightarrow \sqrt{p(1-p)} = 0.5$$
$$p = .4 \Rightarrow \sqrt{p(1-p)} = \sqrt{.24} = .49$$
$$p = .3 \Rightarrow \sqrt{p(1-p)} = \sqrt{.21} = .46$$

SO:

$$1-\alpha = P\left[-z_{\alpha/2} \le \frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \le z_{\alpha/2}\right]$$

ERGO:

$$1-\alpha = P\left[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

SUPPOSE WE PERFORM A BINOMIAL EXPERIMENT WITH n TRIALS AND UNKNOWN PROBABILITY OF "SUCCESS" p ON EACH TRIAL. WE WISH TO TEST $H_0: p = p_0$. IN ORDER TO DO THIS, SET α, & COMPUTE A z STATISTIC:

$$z = \frac{x - p_0 n}{\sqrt{n p_0 (1-p_0)}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \quad ; \hat{p} = \frac{x}{n}$$

IF $H_1: p > p_0$ REJECT H_0 IF $z > z_\alpha$
IF $H_1: p < p_0$ REJECT H_0 IF $z < -z_\alpha$
IF $H_1: p \neq p_0$ REJECT H_0 IF $z > z_{\alpha/2}$
OR $z < -z_{\alpha/2}$

EXAMPLE: (?)

LET p_{SMW} = PROPORTION OF SMW WHO FAVOR NIXON OVER McG.
LET p_R = " " ROSE " " " " "
TAKE RANDOM SAMPLE OF n_{SMW} AND n_R FROM THE TWO POPULATIONS

$H_0: p_{SMW} = p_R$
$H_1: p_{SMW} < p_R$

SUPPOSE WE HAVE A BINOMIAL EXPERIMENT WITH n_1 TRIALS AND UNKNOWN PROBABILITY OF SUCCESS p_1 AND A SECOND BINOMIAL EXPERIMENT WITH PARAMETERS n_2 AND (UNKNOWN) p_2. WE WISH TO TEST $H_0: p_1 = p_2$

1-30-72

(pg 197-8 ; 1, 4, 5 (PROB 5 A LOWER TAIL TEST))

(CONT.)

WE CAN SHOW THAT IF H_0 IS TRUE, THEN

$$Z = \frac{X_1/n_1 - X_2/n_2}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}}$$ IS APPROXIMATELY

A $N(0,1)$ WHEN n_1 AND n_2 IS LARGE.

(X_1, X_2) = NUMBER OF SUCCESSES IN (n_1, n_2) TRIALS

$$\hat{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

PROOF: H_0: $p_1 = p_2 = p$

$$X_1 \sim N(n_1 p, n_1 p(1-p))$$

$$\Rightarrow \begin{cases} X_1/n_1 = N\left(p, \frac{p(1-p)}{n_1}\right) \\ X_2/n_2 = N\left(p, \frac{p(1-p)}{n_2}\right) \end{cases}$$

$$\therefore \frac{X_1}{n_1} - \frac{X_2}{n_2} = N\left(p - p, \frac{p(1-p)}{n_1} + \frac{p(1-p)}{n_2}\right)$$
$$= N\left(0, p(1-p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$$

$$\Rightarrow \frac{X_1/n_1 - X_2/n_2}{\sqrt{p(1-p)(1/n_1 + 1/n_2)}} = N(0,1)$$

NOW

$$\hat{p} = E(p) = E\left[\frac{X_1 + X_2}{n_1 + n_2}\right]$$
$$= \frac{1}{n_1 + n_2}\left[E(X_1) + E(X_2)\right]$$
$$= \frac{pn_1 + pn_2}{n_1 + n_2}$$
$$= p$$

EXAMPLE ON pg 195

SUPPOSE WE PERFORM k BINOMIAL EXPERIMENTS WITH n_j AND PROBABILITY OF SUCCESS P_j ON THE j^{TH} EXPERIMENT $\ni j = 1, 2, 3, \ldots, k$, THEN
$$\chi^2_k = \sum_{j=1}^{k}\left[\frac{(x_j/n_j - P_j)^2}{\frac{P_j(1-P_j)}{n_j}}\right] = \sum_{j=1}^{k}\frac{(x_j - n_jP_j)^2}{n_jP_j(1-P_j)},$$

WHERE x_j IS THE NUMBER OF SUCCESSES IN THE n_j TRIALS IS, APPROXIMATELY FOR $n_1, n_2, n_3, \ldots n_k$ ALL LARGE.
WE COULD USE THIS RESULT TO TEST
$$H_0: P_1 = P_2 = \ldots = P_j = \ldots = P_k = P_0 \ni P_0$$
IS A GIVEN NUMBER. ALWAYS USE UPPER TAIL TEST.
SUPPOSE WE WANT TO TEST
$H_0: p_1 = p_2 = \ldots = P_j = \ldots = P_K$ AGAIST
H_1: NOT ALL THE P_j'S ARE EQUAL, THEN
$$\chi^2 = \sum_{j=1}^{k}\frac{(x_j - n_j\hat{p})^2}{n_j p(1-\hat{p})} \ni \hat{p} = \frac{\sum_{j=1}^{k} x_j}{\sum_{j=1}^{k} n_j} = \frac{x}{n},$$

IS APPROXIMATELY FOR EACH n_j LARGE, A χ^2_{k-1} RANDOM VARIABLE.

LET $e_{1j} = n_j\hat{p} = n_j\frac{x}{n}$, AND
$e_{2j} = n_j\left(1 - \frac{x}{n}\right) = \frac{n_j(n-x)}{n}$
$f_{1j} = x_j$ AND $f_{2j} = n_j - x_j$
FOR $j = 1, 2, 3, \ldots, k$
THEN THE ABOVE χ^2_{k-1} RANDOM VARIABLE CAN BE WRITTEN AS:
$$\chi^2 = \sum_{i=1}^{2}\sum_{j=1}^{k}\frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

Y Proof

2-1-73

$H_0: p_1 = p_2 = \dots = p_j = \dots = p_k$

n_1	f_{11}	
n_2	f_{12}	
$\vdots$	$\vdots$	
n_j	$f_{1j} = x_j$	$f_{2j} = n_j - x_j$
$\vdots$	$\vdots$	
n_k	f_{1k}	
	NUMBER OF SUCCESSES	NUMBER OF FAILURES

TO TEST H_0, SET α, AND COMPUTE

$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{1}{e_{ij}} (f_{ij} - e_{ij})^2$$

$\ni e_{1j} = n_j \underbrace{\frac{x}{n}}_{= \text{ESTIMATE OF SUCCESSES}} \ni x = \sum_j x_j$ AND $n = \sum_j n_j$

$e_{2j} = n_j \left(1 - \frac{x}{n}\right) =$ ESTIMATE OF FAILURES.

WE REJECT H_0 WHEN $\chi^2 > \chi^2_{k-1;\alpha}$

SAMPLE	1	2	$\dots$	j	$\dots$	k	
SUCCESS	f_{11}	f_{21}	$\dots$	f_{1j}	$\dots$	f_{1k}	$x = \sum_j f_{1j}$
FAILURE	f_{21}	f_{22}		f_{2j}		f_{2k}	$n - x = \sum f_{2j}$
n_1	n_1	n_2		n_j		n_k	$n = \sum_j n_j$

$e_{1j} = \frac{n_j x}{n}$

$e_{2j} = \frac{n_j (n - x)}{n}$

EX

	1	2	3	4	
SUC.	29 $\boxed{15.4}$	12 $\boxed{18.2}$	8 $\boxed{14.0}$	21 $\boxed{22.4}$	70
FAIL	81 $\boxed{94.6}$	118 $\boxed{111.8}$	92 $\boxed{86}$	139 $\boxed{137.6}$	430
	110	130	100	160	500

$e_{11} = \frac{(110)(70)}{500} = 15.4$

$e_{12} = \frac{(70)(130)}{500} = 18.2$

CHAPT 6 # 20 198-199 - DO 8, 12, 13, 14

Pg 191 # 8

Pg 196

DO PROB. # 7, 12,

2-2-73

198-9; $(7,12)^{\text{TO HAND IN}}$ 8, 10, 13, 14

LET p_{ij} = PROPORTION OF STUDENTS @ j^{TH} INSTITUTION THAT FAVOR i^{TH} CANDIDATE

<u>H_0:</u> FOR EACH i $(i=1,2,3)$, $P_{i1}=P_{i2}=P_{i3}$

CONTINGENCY TABLES:

WE SHALL NOW DESCRIBE TWO TESTS OF HYPOTHESIS WHICH ARE TESTED IN THE SAME WAY USING THE χ^2 DISTRIBUTION

1) SUPPOSE WE MAKE n_1 INDEPENDENT REPEATED TRIALS OF AN EXPERIMENT WITH OUR MUTUALLY EXCLUSIVE OUTCOMES POSSIBLE ON EACH TRIAL, n_2 INDEPENDENT REPEATED TRIALS OF A SECOND EXPERIMENT WITH r OUTCOMES POSSIBLE ON EACH TRIAL, ..., and n_k INDEPENDENT REPEATED TRIALS OF THE k^{TH} EXPERIMENT WITH r OUTCOMES ON EACH TRIAL. LET THE PROBABILITY OF THE i'TH OUTCOME ON THE j^{TH} EXPERIMENT BE P_{ij} (ALL UNKNOWN.) WE WISH TO TEST:

$$H_0: P_{i1}=P_{i2}=P_{i3}=\ldots=P_{ij}=\ldots=P_{ik}$$

FOR <u>EACH</u> i $(i=1,2,\ldots,r)$

AGAINST ITS DENIAL.

ii) FOR FOUR COLLEGES & THREE CANDIDATES:

$$H_0: \quad P_{11}=P_{12}=P_{13}=P_{14} \qquad k=4$$
$$P_{21}=P_{22}=P_{23}=P_{24} \qquad r=3$$
$$P_{31}=P_{32}=P_{33}=P_{34}$$

NOTE $P_{11}+P_{21}+P_{31}=1$

2) SUPPOSE WE MAKE n INDEPENDENT REPEATED TRIALS OF AN EXPERIMENT IN WHICH THERE ARE $r \cdot k$ POSSIBLE MUTUALLY EXCLUSIVE POSSIBILITIES ON EACH TRIAL. THE OUTCOMES CAN BE CLASSIFIED ACCORDING TO TWO CRITERIA AND THERE ARE r WAYS OF CLASSIFYING AN OUTCOME OF TRIAL ACCORDING TO THE FIRST CRITERIA AND k WAYS OF CLASSIFYING THE OUTCOME ACCORDING TO THE SECOND CRITERION.

LET p_{ij} BE THE PROBABILITY THAT AN OUTCOME OF A TRIAL BELONGS TO THE i^{TH} CLASS OF THE FIRST CRITERION OF CLASSIFICATION AND THE j^{TH} CLASS OF THE SECOND CRITERION OF CLASSIFICATION ($i = 1, 2, \ldots, r$ AND $j = 1, 2, \ldots, k$)

LET $p_{i\cdot} = \sum_{j=1}^{k} p_{ij}$ AND $p_{ij} = \sum_{i=1} p_{ij}$

WE WISH TO TEST

$H_0 : p_{ij} = p_i \, p_{ij}$ FOR ALL (i, j) PAIRS AGAINST ITS DENIAL, ie WE WANT TO TEST THE NULL HYPOTHESIS THE TWO METHODS ARE INDEPENDENT. IN BOTH CASES, THE RESULTS OF THE n TRIALS RESULT IN AN $r \times k$ TABLE, CALLED A CONTINGENCY TABLE. LET f_{ij} BE THE NUMBER OF OBSERVATIONS IN THE i^{TH} ROW AND j^{TH} COLUMN OF THE TABLE. ($i = 1, 2, \ldots, n$). LET $n_{ij} = \sum_{i=1} f_{ij}$, $n_{i\cdot} = \sum_{j=1}^{k} f_{ij}$, AND LET $e_{ij} = n_{i\cdot} \, n_j / n$ FOR EACH (ij) PAIR

2-5-73

FOR EACH f_{ij} REASONABLY LARGE IF EITHER OF THE FOREMENSIONED NULL HYPOTHESIS ARE TRUE:

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

IS APPROX. A χ^2 DISTRIBUTUTION WITH $(r-1)(k-1)$ DEGREES OF FREEDOM.

H_0 IS REJECTED FOR $\chi^2 > \chi^2_{(R-1)(K-1);\alpha}$

GOODNESS OF FIT

SUPPOSE WE HAVE TAKEN A RANDOM SAMPLE OF n FROM AN UNKNOWN POPULATION AND HAVE MADE A FREQ. DISTRIBUTION OF THE RESULTS, WITH FREQUENCY f_i FOR THE i^{TH} CLASS. $(i = 1, 2, 3, \ldots, k)$. WE WANNA TEST H_0: THE DATA HAS COME FROM A POPULATION WITH A CERTAIN POPULATION RANDOM VARIABLE.

LET $\hat{p}_i$ BE THE EXACT OR ESTIMATED PROBABILITY A PIECE OF DATA FALLS IN THE i^{TH} CLASS ASSUMING H_0 IS TRE, AND LET $e_i = n\hat{p}_i$. THEN H_0 IS TRUE IF:

$$\chi^2 = \sum_{i=1}^{k} \frac{(f_i - e_i)^2}{e_i}$$

IS, FOR EACH f_i, SUFFICIENTLY LARGE APP. A χ^2 DISTRIBUTION WITH $K-1$ DEGREES OF FREEDOM (THE NUMBER OF POPULATION PARAMETERS IS UNKNOWN UNDER H_0 WHICH MUST BE ESTIMATED TO COMPUTE $\hat{p}_i$'s)

2-6-73

THE SIGN TEST: SUPPOSE OUR POPULATION RANDOM VARIABLE IS CONTINUOUS WITH A SYMMETRIC DENSITY FUNCTION. WE WISH TO TEST $H_0: \mu = \mu_0$ AGAINST ONE OF THE ALTERNATIVES $H_1: \mu < \mu_0$; $H_1: \mu > \mu_0$; OR $H_1: \mu \neq \mu_0$. SUPPOSE WE SAY WE HAVE A SUCCESS IF A PIECE OF SAMPLE DATA X_i $(i = 1, 2, \ldots, n)$ IS LARGER THAN μ_0. THEN WE CAN TEST H_0 COUNTING THE NUMBER OF SUCCESSES OUT OF THE n TRIALS AND TESTING $H_0: p = 1/2$ AGAINST $H_1: p < \frac{1}{2}$, $H_1 > 1/2$ OR $H_1: p \neq 1/2$ USING THE TEST FOR THE PARAMETER p OF A BINOMIAL DISTRIBUTION.

EX) (IN BOOK)

$H_0: \mu = 28°$; $H_1: \mu \neq 28°$

$H_0: p = 1/2$; $H_1: p \neq 1/2$ ⇐ EQUIVALENT

$n = 19$

$$Z = \frac{x - np_0}{\sqrt{np_0(1-p_0)}} = \frac{18 - 19 \times (\frac{1}{2})}{\sqrt{19 \times \frac{1}{2} \times \frac{1}{2}}}$$

WE CAN COMPARE SUCH A NON-PARAMETRIC TEST WITH THE CORRESPONDING PARAMETRIC TEST BY THE ASYMPTOTIC RELATIVE EFFICIENCY (IF IT EXISTS)

$A.R.E. = \lim_{n\to\infty} \frac{n}{n^*}$ ∋ n^* IS THE SAMPLE SIZE NECESSARY FOR THE NON-PARAMETRIC TEST TO HAVE THE SAME POWER AS THE PARAMETRIC TEST HAS FOR A SAMPLE SIZE OF n.

THE A.R.E. FOR THE SIGN TEST IS 0.63

SUPPOSE WE TAKE RANDOM SAMPLES OF n_1 & n_2 FROM INDEPENDENT CONTINUOUS POPULATIONS 1 & 2. WE WANNA TEST THE NULL HYPOTHESIS:

H_0: THE POPULATIONS ARE IDENTICAL

AGAINST THE ALTERNATIVE

H_1: THEIR MEANS AREN'T IDENTICAL

PARAMETRIC EQUIVALENT:

$$H_0: \mu_1 = \mu_2 \; ; \; H_1: \mu_1 \neq \mu_2$$

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{1/n_1 + 1/n_2}}$$

WE CAN DO THIS BY JOINTLY ARRANGING THE $n_1 + n_2$ OBSERVATIONS FROM THE SMALLEST TO THE LARGEST OBSERVATION, AND ASSIGNING RANKS TO ALL OF THE $n_1 + n_2$ OBSERVATIONS BEGINNING WITH 1 FOR THE SMALLEST AND ENDING WITH RANK $n_1 + n_2$ FOR THE LARGEST. LET R_1 BE THE SUM OF THE RANKS FOR THE n_1 OBSERVATION IN SAMPLE ONE. IF n_1 AND n_2 ARE BOTH MODERATELY LARGE AND H_0 IS TRUE, THEN $U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$ IS $\sim N(\mu, \sigma^2)$ WITH $\mu_u = \frac{n_1 n_2}{2}$ AND

$$\sigma_u^2 = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}$$

2-9-73

THE RESULTING TEST OF HYPOTHESIS IS CALLED THE MANN-WHITNEY μ TEST. IT'S A.R.E. RELATIVE TO THE 2 SAMPLE t TEST IS 0.955.

SUPPOSE WE TAKE RANDOM SAMPLES $n_1, n_2, \ldots, n_k$ FROM INDEPENDENT CONTINUOUS POPULATIONS, $1, 2, \ldots, k$. WE WANNA TEST THE NULL HYPOTHESIS THAT THE POPULATIONS ARE IDENTICAL AGAINST THE ALTERNATIVE THAT THEIR MEANS ARE NOT EQUAL. WE CAN DO THIS BY JOINTLY ARRANGING ALL $\sum_{i=1}^{k} n_i$ OBSERVATIONS FROM THE SMALLEST TO THE LARGEST AND ASSIGNING RANKS TO ALL $\sum_{i=1}^{k} n_i$ OBSERVATIONS BEGINNING WITH 1 FOR THE SMALLEST AND ENDING WITH $\sum_{i=1}^{n} n_i$ FOR THE LARGEST. LET R_i $(i = 1, 2, \ldots, k)$ BE THE SUM OF RANKS FOR THE n_i OBSERVATIONS IN THE SAMPLE FROM THE i^{TH} POPULATION. IF ALL THE n_i'S ARE REASONABLY LARGE AND H_0 IS TRUE, THEN

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(n+1) \ni n = \sum_{i=1}^{k} n_i$$

IS APPROXIMATELY A χ^2_{k-1} RANDOM VARIABLE. USE AN UPPER TAIL TEST. THE A.R.E. IS 0.955

(KRUSKAL-WALLIS TEST)

SUPPOSE WE TAKE A SAMPLE OF n FROM A POPULATION. WE WISH TO TEST H_0: THE SAMPLE IS RANDOM AGAINST THE ALTERNATIVE IT ISN'T. WE CAN USE THE RUN TEST FOR THAT PURPOSE. SUPPOSE WE HAVE TWO TYPES OF SYMBOLS, SAY S & F. A RUN IS A SUCCESSION OF IDENTICAL SYMBOLS. SUPPOSE THERE ARE n_1 S'S AND n_2 F'S IN THE DATA AS IT WAS TAKEN. LET μ BE THE NUMBER OF RUNS IN THE n_1 S'S AND n_2 F'S. THEN IF n_1 & n_2 ARE BOTH LARGE (>9), μ IS APPROXIMATELY NORMALLY DISTRIBUTED WITH MEAN

$$\mu_\mu = \frac{2n_1 n_2}{n_1+n_2} + 1$$

AND VARIANCE

$$\sigma_\mu^2 = \frac{2n_1 n_2(2n_1 n_2 - n_1 - n_2)}{(n_1+n_2)^2(n_1+n_2-1)}$$

<u>ASSIGN:</u> Pp. 217-8; 2, 6, 7, 10 (HAND IN)

Pp. 223-4 ; 2, 4, 5, 6

2·9·73

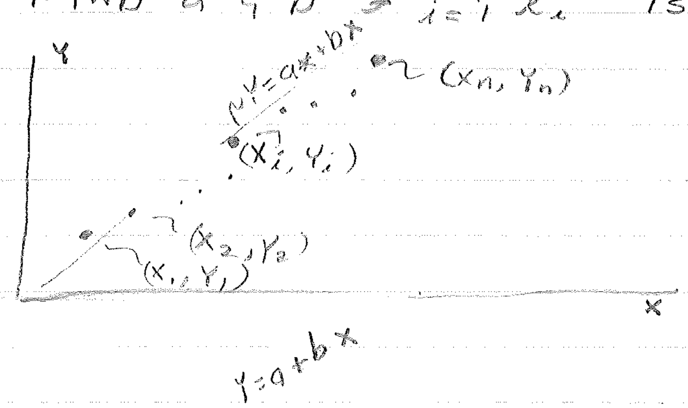
REGRESSION AND CORRELATION

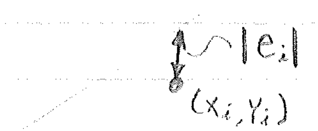
LET X BE A NON-RANDOM VARIABLE & LET Y BE A RANDOM VARIABLE $\ni$ $Y = \alpha + \beta X + \epsilon$, WHERE $E(\epsilon) = 0$, (i.e. $E(Y) = \alpha + \beta X$). SUPPOSE WE DO NOT KNOW α & β BUT WISH TO ESTIMATE THEM USING n PEARS OF DATA $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ WHERE $x_1, x_2, \ldots, x_n$ ARE CHOSEN VALUES AND $Y_1, Y_2, \ldots, Y_n$ ARE THE CORRESPONDING OUTCOMES OF Y. WE CAN DO THIS BY FITTING THE LEAST SQUARE STRAIGHT LINE TO THE n PEARS OF DATA AND USE THE SLOPE AND Y INTERCEPT OF THE RESULTING LINE TO ESTIMATE β AND α RESPECTIVELY. THIS WILL BE DONE AS FOLLOWS: (LEAST SQUARES FIT)

LET $Y_i = a + b x_i + e_i$

WE WISH TO FIND a & b $\ni$ $\sum_{i=1}^{n} e_i^2$ IS MINIMUM.

EX)





NOW, WE GOTTA MINIMIZE Σe_i^2 WITH RESPECT TO a & b

$$\sum_{i=1}^{n} e_i^2 = S = \sum_{i=1}^{n} (Y_i - a - bx_i)^2$$

$$\frac{\delta S}{\delta a} = \sum_{i=1}^{n} -2(Y_i - a - bx_i) = 0 \qquad (A)$$

OR $\sum_{i=1}^{n} Y_i - na - b\sum_{i=1}^{n} X_i = 0$

$$\frac{\delta S}{\delta b} = \sum_{i=1}^{n} 2(Y_i - a - bx_i)(-x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} X_i Y_i - a\Sigma X_i - b\Sigma X_i^2 = 0 \qquad (B)$$

FROM (A); $a = \frac{\Sigma Y_i}{n} - b\frac{\Sigma X_i}{n}$

INTO (B) $\Sigma X_i Y_i - \left(\frac{1}{n}\Sigma Y_i - \frac{b}{n}\Sigma X_i\right)(\Sigma X_i) - b\Sigma X_i^2 = 0$

$\bar{Y} = \Sigma Y_i / n$; $\bar{X} = \Sigma X_i / n$

$$\Sigma X_i Y_i - n\bar{X}\bar{Y} + nb\bar{X}^2 - b\Sigma X_i^2$$

$$\Rightarrow b = \frac{\Sigma X_i Y_i - n\bar{X}\bar{Y}}{\Sigma X_i^2 - n\bar{X}^2} = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2}$$

2-12-73

WE NOW HAVE THE LEAST SQUARE EQUATION:

$$Y = a + bX$$

WHERE

$$b = \frac{\Sigma X_i Y_i - n\bar{X}\cdot\bar{Y}}{\Sigma(X_i^2 - n\bar{X}^2)}$$

$$a = \bar{Y} - b\bar{X}$$

TO SHOW THAT THE TWO EQ. FOR b ARE EQUIVALENT

NUMERATER: $\Sigma(Y_i - \bar{Y})(X_i - \bar{X})$

$$= \Sigma Y_i X_i - \bar{Y}\Sigma X_i - \bar{X}\Sigma Y_i + \Sigma\bar{Y}\bar{X}$$

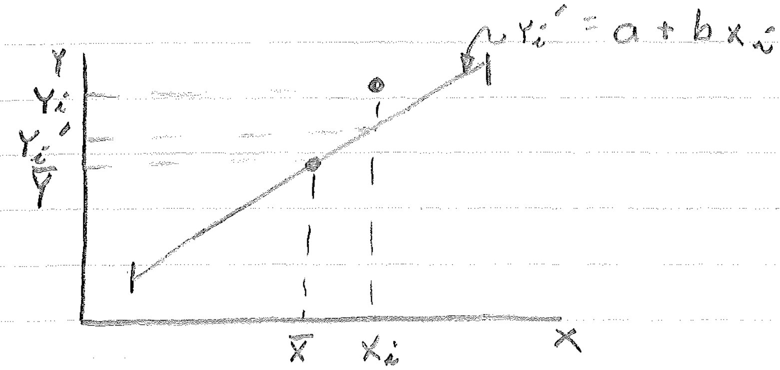
$$= \Sigma Y_i X_i - n\bar{Y}\bar{X} - n\bar{X}\bar{Y} + n\bar{X}\bar{Y}$$

$$= \Sigma Y_i X_i - n\bar{Y}\bar{X}$$

DENOMINATOR: $\Sigma(X_i - \bar{X})^2 = \Sigma X_i^2 - 2\bar{X}\Sigma X_i + \Sigma\bar{X}^2$

$$= \Sigma X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$$

$$= \Sigma X_i^2 - n\bar{X}^2$$



$$Y_i - \bar{Y} = (Y_i - Y_i') + (Y_i' - \bar{Y})$$

<u>THEOREM:</u> LET

1) $SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$; DEVIATION ABOUT $(\bar{X}, \bar{Y})$

2) $SSD = \sum (Y_i - Y_i')^2$; DEVIATION ABOUT PREDICTED POINT

3) $SSR = \sum (Y_i' - \bar{Y})^2$; DEVIATION DUE TO REGRESSION OR LEAST SQUARE LINE

THEN $SST = SSD + SSR$

PROOF:

$$Y_i - \bar{Y} = (Y_i - Y_i') + (Y_i' - \bar{Y})$$

$$(Y_i - \bar{Y})^2 = (Y_i - Y_i')^2 + (Y_i' - \bar{Y})^2 + 2(Y_i - Y_i')(Y_i' - \bar{Y})$$

$$\sum (Y_i - \bar{Y})^2 = \sum (Y_i - Y_i')^2 + \sum (Y_i' - \bar{Y})^2 + 2\sum (Y_i - Y_i')(Y_i - \bar{Y})$$

$$\Rightarrow SST = SSD + SSR + 2\sum (Y_i - Y_i')(Y_i' - \bar{Y})$$

BUT THIRD TERM IS ZERO (SHOW)

RECALL: $a = \bar{Y} - b\bar{X}$ & $Y_i' = a + bx_i$

$$b = \frac{\sum (x_i - \bar{x})(Y_i - \bar{Y})}{\sum (x_i - \bar{x})^2}$$

$$\Rightarrow \sum (Y_i - Y_i')(Y_i' - \bar{Y}) = \sum (Y_i - a - bx_i)(a + bx_i - \bar{Y})$$

$$= \sum (Y_i - \bar{Y} + b\bar{x} - bx_i)(\bar{Y} - b\bar{X} + bx_i - \bar{Y})$$

$$= \sum \left[b(Y_i - \bar{Y})(x_i - \bar{x}) - b^2 (x_i - \bar{x})^2 \right]$$

$$= b\left[\sum (Y_i - \bar{Y})(x_i - \bar{X}) - b\sum (x_i - \bar{X})^2 \right]$$

BUT $b\sum (x_i - \bar{X})^2 = \sum (Y_i - \bar{Y})(x_i - \bar{X})$

$$\Rightarrow \sum (Y_i - Y_i')(Y_i' - Y) = 0$$

2-13-73

THEOREM: $SSR = b^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$

PROOF: $SSR = \sum_{i=1}^{n} (Y_i' - \bar{Y})^2$

$$= \sum (a + bx_i - \bar{Y})^2$$
$$= \sum (\bar{Y} - b\bar{x} + bx_i - \bar{Y})^2$$
$$= \sum (bx_i - b\bar{x})^2$$
$$= b^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

WE WISH NOW TO TEST HYPOTHESIS AND MAKE CONFIDENCE INTERVAL STATEMENTS ABOUT α AND β. WE SHALL MAKE THE ASSUMPTION THAT $Y_i = \alpha + \beta x_i + \varepsilon_i$ $(i = 1, 2, \ldots, n)$ ARE NORMALLY AND INDEPENDENTLY DISTRIBUTED, WITH MEANS $E[Y_i] = \alpha + \beta x_i$ AND COMMON VARIANCE σ^2. LET a & b BE THE LEAST SQUARES ESTIMATE OF α AND β, AND LET $S_e^2 = SSD/n-2$. UNDER THESESE ASSUMPTIONS WE HAVE THE FOLLOWING THEOREM:

THEOREM:

1) $E[a] = \alpha$
2) $E[b] = \beta$
3) $E[S_e^2] = \sigma^2$
4) $\sigma_b^2 = \sigma^2 / \sum (x_i - \bar{x})^2$ = VARIANCE OF b
5) $\sigma_a^2 = \sigma^2 \sum x_i^2 / n \sum (x_i - \bar{x})^2$ = VARIANCE OF a
6) $T = \frac{a - \alpha}{S_a}$ AND $T = \frac{b - \beta}{S_b}$ ARE STUDENT t R.V.'S WITH $n-2$ DEGREES OF FREEDON

$$\ni \; S_a^2 = \frac{S_e^2 \sum x_i^2}{n \sum (x_i - \bar{x})^2} \; ; \; S_b^2 = \frac{S_e^2}{\sum (x_i - \bar{x})^2}$$

THE $1-\alpha$ CONFIDENCE INTERVAL FOR α IS

$$a \pm t_{n-2;\alpha/2}\, S_a$$

THE $1-\alpha$ CONFIDENCE INTERVAL FOR β IS

$$b \pm t_{n-2;\alpha/2}\, S_b$$

FOR $H_0: \beta = b_0$

SET α

COMPUTE $T = \dfrac{b - b_0}{S_b}$

$$\begin{cases} \text{IF } H_1: \beta > b_0, \text{ REJECT } H_0 \text{ FOR } T > t_{n-2;\alpha} \\ \text{"} \quad H_1: \beta < b_0, \quad \text{"} \quad \text{"} \quad \text{"} \quad T < -t_{n-2;\alpha} \\ \text{"} \quad H_1: \beta \neq b_0, \quad \text{"} \quad \text{"} \quad \text{"} \quad T > t_{n-2;\alpha/2} \\ \qquad \text{OR } T < -t_{n-2;\alpha/2} \end{cases}$$

FOR $H_0: \alpha = a_0$

SET α

COMPUTE $T = \dfrac{a - a_0}{S_a}$

$$\begin{cases} \text{IF } H_1: \alpha > a_0, \text{ REJECT } H_0 \text{ FOR } T > t_{n-2;\alpha} \\ \text{"} \quad H_1: \alpha < a_0, \quad \text{"} \quad \text{"} \quad \text{"} \quad T < -t_{n-2;\alpha} \\ \text{"} \quad H_1: \alpha \neq a_0, \quad \text{"} \quad \text{"} \quad \text{"} \quad T > t_{n-2;\alpha/2} \\ \qquad \text{OR } T < -t_{n-2;\alpha/2} \end{cases}$$

THEOREM:

IF X_0 IS A GIVEN VALUE OF X, THEN

$$T = \frac{a + bX_0 - (\alpha + \beta X_0)}{S_e\sqrt{\dfrac{1}{n} + \dfrac{(X_0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}}$$

HAS A STUDENT'S t DISTRIBUTION WITH $n-2$ D.O.F.. THIS LEADS TO THE $1-\alpha$ CONFIDENCE INTERVAL FOR $E[Y] = \alpha + \beta X_0$ OF

$$a + bX_0 \pm t_{n-2;\alpha/2}\, S_e\sqrt{\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}$$

THEOREM:

IF X_0 IS A GIVEN VALUE OF X, THEN

$$T = \frac{Y - a - b X_0}{S_e \sqrt{1 + \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}}$$

HAS A STUDENT'S t DISTRIBUTION WITH $n-2$ DEGREES OF FREEDOM. THIS LEADS TO THE $1-\alpha$ LIMIT OF PREDICTION FOR Y WHEN $X = X_0$ OF:

$$a + b X_0 \pm t_{n-2;\alpha/2} S_e \sqrt{1 + \frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\Sigma(X_i - \bar{X})^2}}$$

EXAMPLE

X	20	30	40	60	70	90	100	120	150	180
Y	3.5	7.4	7.1	15.6	11.1	14.9	23.5	27.1	22.1	32.9

$\Sigma X_i = 860 \qquad \Sigma Y_i = 165.2$

$\Sigma X_i^2 = 98800 \qquad \Sigma Y_i^2 = 3563.48$

$\Sigma X_i Y_i = 18,469$

$\Rightarrow \bar{X} = \frac{\Sigma X_i}{n} = \frac{860}{10} = 86 \; ; \; \bar{Y} = \frac{\Sigma Y_i}{n} = 16.52$

$\Sigma(X_i - \bar{X})^2 = \Sigma X_i^2 - n\bar{X}^2 = 24840$

$SST \triangleq \Sigma(Y_i - \bar{Y})^2 = \Sigma Y_i^2 - n\bar{Y}^2 = 833$

$\Sigma(X_i - \bar{X})(Y_i - \bar{Y}) = \Sigma X_i Y_i - n\bar{X}\bar{Y} = 4259$

$b = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2} = 0.171$

$a = \bar{Y} - b\bar{X} = 1.90$

$\Rightarrow Y = 1.90 + 0.171 X$

$SSR = b^2 \Sigma(X_i - \bar{X})^2 = 729$

$SSD = SST - SSR = 140$

$S_e^2 = \frac{SSD}{n-2} = \frac{104}{8} = 13$

2-19-73

MULTIPLE REGRESSION

LET $X_1, X_2, \ldots, X_p$ BE VARIABLES WHICH ARE NOT RANDOM AND LET Y BE A RANDOM VARIABLE $\ni$ $Y = B_0 + B_1 X_1 + B_2 X_2 + \ldots + B_r X_r + \epsilon$, WHERE $E(\epsilon) = 0$, THAT IS

$E[Y] = B_0 + B_1 X_1 + B_2 X_2 + \ldots + B_r X_r$.

SUPPOSE $B_0, B_1, B_2, \ldots, B_r$ ARE NOT KNOWN, BUT WE WISH TO ESTIMATE THEM USING DATA $\{(X_{11}, X_{21}, \ldots, X_{r1}, Y_1), (X_{12}, X_{22}, \ldots, X_{r2}, Y_2), \ldots, (X_{1n}, X_{2n}, \ldots, X_{rn}, Y_n)\}$, $\ni$ Y_j IS THE OUTCOME OF Y CORRESPONDING TO THE CHOSEN VALUES OF $X_{1j}, X_{2j}, \ldots, X_{rj}$, FOR $j = 1, 2, \ldots, n$. WE CAN DO THIS BY "FITTING THE HYPERPLANE"

$$Y = b_0 + b_1 X_1 + b_2 X_2 + \ldots + b_r X_r$$

TO THE DATA BY THE FOLLOWING METHODS.

LET $Y_j' = b_0 + b_1 X_{1j} + b_2 X_{2j} + \ldots + b_2 X_{ij}$ FOR $j = 1, 2, \ldots, n$. CHOOSE $b_0, b_1, \ldots b_r$ SUCH THAT

$$S = \sum_{j=1}^{n} (Y_j - Y_j')^2 \quad \text{IS MINIMUM}$$

THESE MINIMIZING VALUES OF $b_0, b_1, \ldots, b_r$ WILL BE USED AS PT. ESTIMATES OF $B_0, B_1, \ldots, B_r$ RESPECTIVELY.

$$S = \sum_{j=1}^{n} (Y_j - b_0 - b_1 X_{1j} - b_2 X_{2j} - \ldots - b_i X_{ij} - \ldots - b_i X_{ij})^2$$

TAKING PARTIAL DERIVATIVES OF S WITH RESPECT TO $b_0, b_1, \ldots, b_r$ AND THEN SETTING THEM EQUAL TO 0, WE GET

$$\frac{\delta S}{\delta b_0} = 2\sum_{j=1}^{n} (Y_j - b_0 - b_1 X_{1j} - b_{2j} - \ldots - b_i X_{ij} - \ldots - b_2 X_{ij})(-1) = 0$$

AND

$$\frac{\delta S}{\delta b_i} = 2\sum_{j=1}^{n}(Y_j - b_0 - b_1 x_{1j} - b_2 x_{2j} - \dots - b_i X_{ij} - \dots b_r X_{rj}) X_{ij}$$

THIS GIVES THE EQUATIONS:

$$b_0 n + b_1 \sum_j X_{1j} + b_2 \sum_j X_{2j} + \dots + b_i \sum_j X_{ij} + b_r \sum_j X_{rj} = \sum Y_j$$

$$b_0 \sum_j X_{ij} + b_1 \sum_j X_{1j} X_{ij} + \sum_j X_{2j} X_{ij} + \dots + b_i \sum_j X_{ij}^2 + \dots + b_r \sum_j X_{ij} X_{rj} = \sum_j X_{ij} Y_j$$

FOR $i = 1, 2, \dots, r$

LET
$$\underline{X} = \begin{bmatrix} 1 & x_{11} & x_{21} & \dots & X_{i1} & \dots & X_{r1} \\ 1 & x_{12} & x_{22} & \dots & X_{i2} & \dots & X_{r2} \\ \vdots \\ 1 & X_{1j} & X_{2j} & \dots & X_{ij} & \dots & X_{rj} \\ \vdots \\ 1 & X_{1n} & X_{2j} & \dots & X_{in} & \dots & X_{rn} \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_j \\ \vdots \\ Y_n \end{bmatrix}$$

$$\underline{X^T X} = \begin{bmatrix} n & \sum X_{1j} & \sum X_{2j} & \dots & \sum X_{ij} & \dots & \sum X_{rj} \\ \sum X_{1j} & \sum X_{1j}^2 & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix}$$

$$B = \begin{bmatrix} b_0 \\ b_1 \\ b_i \end{bmatrix}$$

$$X^T X B = X^T Y \Rightarrow B = (X^T X)^{-1} X^T Y$$

NOW: $\frac{b_i - B_i}{S_{b_i}}$ IS A t STATISTIC

POLYNOMIAL REGRESSION: LET X BE A NON RANDOM VARIABLE AND LET Y BE A RANDOM VARIABLE $\Rightarrow$

$Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \ldots + \beta_r X^r + e \quad \ni E(e) = 0$

$\therefore \; E[Y] = \beta_0 + \beta_1 X + \ldots \beta_r X^r$

SUPPOSE WE DO NOT KNOW $\beta_0, \ldots, \beta_r$, BUT WISH TO ESTIMATE THEM USING THE n PAIRS OF DATA

$(x_1, Y_1), (x_2, Y_2), \ldots (x_n, Y_n)$ $(r < n)$

WE SHALL DO THIS BY USING THE LEAST SQUARE TECHNIQUES TO FIT $Y = b_0 + b_1 X + b_2 X^2 + \ldots + b_r X^r$ TO THE DATA AND USING $b_0, b_1, \ldots, b_r$ AS POINT ESTIMATES OF $\beta_0, \beta_1, \ldots, \beta_r$.

THIS CAN BE DONE BY LETTING $X_{ij} = X_j^i$

$x_j \quad x_j^2 \quad x_j^3 \ldots + \ldots x_j^4 \ldots x_n^r \ldots Y_j$

Pg 249-254, 1, 2, 5, 8, 12, 13, 14

2-20-73

CORRELATION

DEFINITION: LET X AND Y HAVE JOINT DENSITY FUNCTION

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}{2(1-\rho)}}$$

FOR $-\infty < x < +\infty$ AND $-\infty < y < \infty$ WHERE $\mu_1, \mu_2, \sigma_1, \sigma_2$, AND ρ ARE CONSTANTS SUCH THAT $-\infty < \mu_1 < \infty$, $-\infty < \mu_2 < \infty$, $\sigma_1 > 0$, $\sigma_2 > 0$, $-1 \leq \rho \leq 1$. THEN $f(x,y)$ IS A BIVARIANT DISTRIBUTION

DEFINITION: LET X AND Y HAVE JOINT DENSITY FUNCTION $f(x,y)$. THEN THE CORRELATION OF X AND Y IS GIVEN BY

$$\rho_{xy} = \frac{E[(X-\mu_x)(X-\mu_y)]}{\sigma_x\sigma_y} = \frac{1}{\sigma_x\sigma_y}\int_{-\infty}^{\infty}\int (x-\mu_x)(x-\mu_y)f(x,y)\,dx\,dy$$

WHERE μ_x AND μ_y ARE THE MEANS ON X AND Y AND σ_x AND σ_y ARE THE STANDARD DEVIATIONS OF X AND Y

THEOREM: $-1 \leq \rho_{xy} \leq 1$

DEFINITION: LET X AND Y HAVE JOINT DENSITY FUNCTION $f(x,y)$. THEN THE CONDITIONAL DENSITY FUNCTION OF Y GIVEN $X = x$ IS $f(y/x) = f(x,y)/f(x)$.

ALSO $E[Y/X=x] = \int_{-\infty}^{\infty} y f(y/x)\,dy$ AND

VARIANCE $(Y/X=x) = \int_{-\infty}^{\infty} [y - E[Y/X=x]]^2 f(y/x)\,dy$.

THEOREM: THE FOLLOWING RESULTS CAN BE PROVED ABOUT THE BIVARIATE NORMAL DISTRIBUTION

1) $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$ WHERE $f_X(x)$ IS THE MARGINAL DENSITY FUNCTION OF X

2) $f_Y(Y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{Y-\mu_2}{\sigma_2}\right)^2}$ WHERE $f_Y(Y)$ IS THE MARGINAL DENSITY FUNCTION OF Y

3) $E(X) = \mu_1$; $E(Y) = \mu_2$
$VAR(X) = \sigma_1^2$; $VAR(Y) = \sigma_2^2$
$\rho_{XY} = \rho$

4) $f(\mu/X = x)$ IS A NORMAL DISTRIBUTION WITH MEAN $E[Y/X = x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1)$ AND VARIANCE $VAR[Y/X = x) = \sigma_2^2(1-\rho^2)$

IMPORTANT REMARKS:

a) $E[Y/X = x] = \alpha + \beta x \ni \alpha$ & β ARE CONSTANT
b) $VAR[Y/X = x] = \sigma_2^2(1-\rho^2) = \sigma^2$, INDEP. OF x
c) WE CAN MAKE POINT AND INTERVAL ESTIMATES ABOUT α, β, σ^2 & $E[Y/X = x]$ JUST AS WE DID WHEN WE ASSUMED EARLIER THAT X WAS NOT A RANDOM VARIABLE.

HOWEVER WE ARE NOW ALSO INTERESTED IN ESTIMATING AND TESTING HYPOTHESES ABOUT ρ.

DEFINITION: SUPPOSE WE TAKE n PAIRS OF DATA $(x_1, Y_1), (x_2, Y_2), \ldots, (x_n, Y_n)$ FROM A BI-VARIANT POPULATION. THEN

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(Y_i - \bar{Y})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$ IS THE SAMPLE CORRELATION COEFFICIENT

NOTE $\sum(x_i - \bar{x})(Y_i - \bar{Y}) = \sum x_i Y_i - n\bar{x}\bar{Y}$
$\sum(x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$

(r IS PT. ESTIMATE OF ρ)

$SST = SSR + SSD$

$SST = \Sigma(Y_i - \bar{Y})^2$

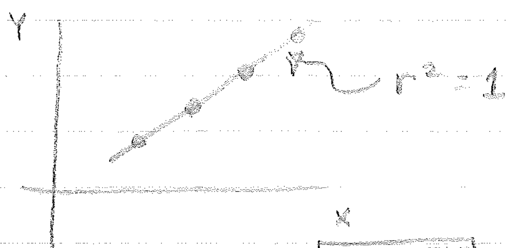
$SSR = \Sigma(Y_i' - \bar{Y})^2 = b^2\,\Sigma(X_i - \bar{X})^2$

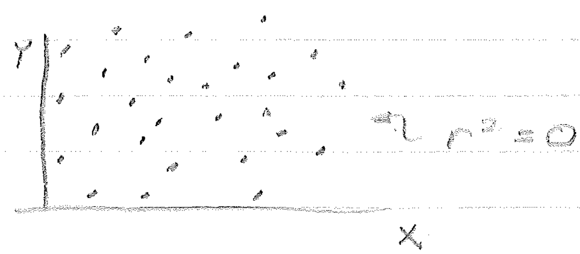
$$r^2 = \frac{[\Sigma(X_i - \bar{X})(Y_i - \bar{Y})]^2}{(\Sigma(X_i - \bar{X})^2)(\Sigma(Y_i - \bar{Y})^2)}$$

$$= \left[\frac{[\Sigma(X_i - \bar{X})(Y_i - \bar{Y})]^2}{\Sigma(X_i - \bar{X})^2}\right]^2 \frac{\Sigma(X_i - \bar{X})^2}{\Sigma(Y_i - \bar{Y})^2}$$

$$= b^2 \; \frac{\Sigma(X_i - \bar{X})^2}{\Sigma(Y_i - \bar{Y})^2}$$

$$= \frac{SSR}{SST}$$





NOW $\quad r = b\,\dfrac{\sqrt{\Sigma(X_i - \bar{X})^2}}{\sqrt{\Sigma(Y_i - \bar{Y})^2}}$

r HAS SAME SIGN AS b

r^2 MEASURES ONLY THE LINEAR RELATIONSHIPS OF X AND Y

THEOREM: IF WE TAKE A RANDOM SAMPLE OF n FROM A BIVARIANT NORMAL DISTRIBUTION, THEN FOR n LARGE $\frac{1}{2}\ln\frac{1+r}{1-r}$ IS APPROXIMATELY NORMALLY DISTRIBUTED WITH MEAN $\frac{1}{2}\ln\frac{1+\rho}{1-\rho}$ AND VARIANCE $\frac{1}{n-3}$

PP 258-9; 1, 3, 8, 4 ← HAND IN 8 AND HAND IN DERIVATION OF CONFIDENCE INTERVAL IN 3

(COMPUTE CORRELATION COEFFICIENT ON FINAL)

2-21-73

Pg 258 # 1

x	Y	x	Y
4.1	13	5.1	22
4.9	12	4.5	21
4.4	11	5.1	13
4.7	10	3.0	37
5.1	13	4.8	13
5.0	14	4.2	19
4.7	21	5.2	12
4.6	14	5.5	14
3.6	26	5.2	21
4.9	25	4.4	29

$\Sigma x_i = 93$; $\Sigma x_i^2 = 439.14$ $\quad \bar{X} = 4.65$

$\Sigma Y_i = 360$; $\Sigma Y_i^2 = 7452$ $\quad \bar{Y} = 18$

$\Sigma x_i Y_i = 1625.4$

$$r^2 = \frac{[\Sigma(x_i - \bar{X})(Y_i - \bar{Y})]^2}{\Sigma(X_i - \bar{X})^2 \Sigma(Y_i - \bar{Y})^2} = \frac{[\Sigma x_i Y_i - n\bar{X}\bar{Y}]^2}{[\Sigma x_i^2 - n\bar{x}^2][\Sigma Y_i^2 - n\bar{Y}^2]}$$

$$\frac{2361.96}{(6.96)(972)} \approx 0.363 \Rightarrow r = \pm 0.603$$

OR:

$$b = \frac{\Sigma(x_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2} = \frac{\Sigma x_i Y_i - n\bar{x}\bar{Y}}{\Sigma x_i^2 - n\bar{x}^2} \Rightarrow \frac{-48.6}{6.69}$$

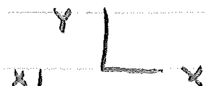
$$= -7.26$$

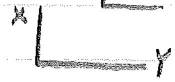
$$\Rightarrow r = b^2 \Sigma(x_i - \bar{x})^2 / \Sigma(Y_i - \bar{Y})^2$$

$$= \frac{(52.7)(6.69)}{972} = 0.363$$

$$\Rightarrow r = -0.602$$

(r HAS THE SAME SIGN AS b)

NOTE: $b<0$ FOR 

$\Rightarrow b<0$ FOR 

HOWEVER, WHEN USING ONE VARIABLE AS INDEPENDENT TO FIND b, BE SURE TO KEEP x AS IN THE EQN.'S

SO $r = -0.6$

b) $H_0: \rho = 0 \qquad \alpha = 0.05$

$$Z = \frac{\frac{1}{2}\left(\ln \frac{1+r}{1-r} - \ln \frac{1+\rho}{1-\rho}\right)}{1/\sqrt{n-3}}$$

$$= \frac{1}{2}\left[\ln \frac{0.4}{1.6} - \cancelto{0}{\ln 1}\right]\sqrt{17}$$

$$= -2.86$$

$-Z_{0.025} = -1.96$

$\Rightarrow$ REJECT H_0

3) CONSTRUCT A 0.95 CONF. INTERVAL FOR $\rho \Rightarrow \alpha = 0.05$

$$\frac{1+r-(1-r)\,e^{\pm 2 Z_{\alpha/2}/\sqrt{n-3}}}{1+r+(1-r)\,e^{\pm 2 Z_{\alpha/2}/\sqrt{n-3}}}$$

$$= \frac{0.4 - 1.6\,e^{\pm(2)(1.96)/\sqrt{17}}}{0.4 + 1.6\,e^{\pm(2)(1.96)/\sqrt{17}}}$$

FOR + SIGN $\Rightarrow \frac{-3.74}{4.54} = -0.82$

FOR − SIGN $\Rightarrow \frac{-0.218}{1.018} = -0.21$

$\Rightarrow \rho$ IS 'TWIXT -0.82 AND -0.21

ANALYSIS OF VARIANCE

SUPPOSE WE TAKE R.S.'S OF n FROM INDEPENDENT POP'S, 1, 2, ..., K. EACH OF WHICH HAS MEAN $\mu_i (i = 1, 2, \ldots, k)$ AND COMMON VARIANCE σ^2. WE WISH TO MAKE POINT AND INTERVAL ESTIMATES OF THE μ_i AND σ^2 AND MOST IMPORTANT WISH TO TEST $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$ AGAINST THE ALTERNATIVE THAT NOT ALL THE μ_i'S ARE EQUAL. IF WE LET Y_{ij} BE THE j^{TH} OBSERVATION FROM THE i^{TH} POPULATION; WE CAN WRITE THE MODEL $Y_{ij} = \mu_i + \epsilon_{ij}$ $\forall i = 1, 2, \ldots, k$ AND $\forall j = 1, 2, 3, \ldots, n$. WE ORDINARILY LET $\mu_i = \frac{1}{k} \sum_{i=1}^{k} \mu_i$ AND LET $\alpha_i = n_i - \mu$ TO GET THE ALTERNATIVE MODEL $Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$ WHERE EACH ϵ_{ij} IN $N(0, \sigma^2)$ RV. AND ALL ϵ_{ij}'S ARE INDEPENDENT, AND THE $\sum_{i=1}^{k} \alpha_i = 0$

$\mu_i \Rightarrow$ OVERALL EFFECT

$\alpha_i \Rightarrow$ EFFECT FROM EACH POPULATION

THE TEST OF HYPOTHESIS $H_0: \mu_1 = \mu_2 = \ldots = \mu_k$

LET $Y_i = \sum_{j=1}^{n} Y_{ij} / n$ BE THE MEAN OF THE i^{TH} SAMPLE AND LET

$$\bar{Y}_{\cdot} = \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{Y_{ij}}{nk} \quad \text{BE THE GRAND MEAN}$$

WE CAN SUBDIVIDE THE TOTAL SUM OF SQUARES OF THE Y_{ij}'S:

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y})^2$$

INTO A SUM OF SQUARES WITHIN SAMPLES

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_i)^2$$

AND A SUM OF SQUARES 'TWIXT SAMPLES

$$SST_r = n \sum_{i=1}^{k} (\bar{Y}_i - \bar{Y})^2$$

<u>THEOREM</u>: $SST = SSE + SST_r$

<u>THEOREM</u>: UNDER ANOVA ASSUMTIONS

1) SST_r + SSE ARE INDEPENDENT
2) $\frac{SSE}{\sigma^2}$ IS A $\chi^2_{k(n-1)}$ R.V. AND IF H_0 IS TRUE $\frac{SSTr}{\sigma^2}$ IS A χ^2_{k-1} R.V.

THUS, IF H_0 IS TRUE $\frac{SSTr}{\sigma^2}$ IS A χ^2_{k-1} RANDOM VARIABLE, AND IF H_0 IS TRUE:

$$\left[\frac{SSTr}{k-1}\right] \Big/ \left[\frac{SSE}{k(n-1)}\right]$$

IS A $F_{k-1;\,k(n-1)}$ RANDOM VARIABLE

(UPPER TAIL TEST)

2-22-72

$$\hat{\sigma}^2 = \frac{\Sigma(Y_i - Y_i')^2}{d\,f_i} = S_e^2$$

DEGREES OF FREEDOM FOR LINEAR: $n-2$

" " " QUADRATIC: $n-3$

" " " FOR K^{TH} ORDER: $n-1-k$

Pg 250, #2 a) $87.9(0.96)^x$

b) 69

#8: $Y = 10.8 + 2.4X - 0.0053X^2$

#14: $Y = 2.22 + 0.22X_1 + 0.06X_2$

205 #8: $\chi^2 = 13.4$

#9: $\chi^2 = 2.3$

THE FINAL

CLOSED BOOK PORTION

BINOMIAL → NORMAL (CENTRAL LIMIT THEM.)

t SAMPLING PLUGS & SUCH (CHAPT. 7 & 8)

(χ^2, t, Z, F)

CONTINGINCY TABLES, & χ^2

NO STATISTICS FROM CHAPT. 11, 12, 13

COMPUTE MEAN, VARIANCE, MEDIAN, ETC.

CURVE FITTING → LEAST SQUARES LINE

COMPUTE CORRELATION COEFFICIENT

ANOVA TABLE (ANALYSIS OF VARIANCE)

SOURCE	D.F	S.S. (SUM OF SQUARES)	MS (MEAN SQUARE)	F
(BETWEEN SAMPLES) TREATMENTS	$k-1$	$SS(T_r)$	$\frac{SS(T_r)}{k-1} = MS(T_R)$	$\frac{M.S.(T_r)}{M.S.(E)}$
(WITHIN SAMPLES) ERROR	$k(n-1)$	SSE	$\left[\frac{SSE}{k(n-1)}\right] = MS(E)$	(REJECT IF F IS TO LARGE)

$$E[M.S.(E)] = \sigma^2$$

$$E[\bar{Y}_i - \bar{Y}] = \mu_i - \mu$$

$\bar{Y}_i \pm t_{k(n-1)} \frac{\sqrt{M.S.E.}}{\sqrt{n}}$ IS A $1-\alpha$ CONFIDENCE INTERVAL FOR μ_i

FOR $\mu_i - \mu_j$: $\bar{Y}_i - \bar{Y}_j \pm t_{k(n-1);\frac{\alpha}{2}} \frac{\sqrt{2M.S.E}}{\sqrt{n}}$

COMPUTATIONAL FORMS

THEM: LET $T_i = \sum_{j=1}^{n} Y_{ij}$ AND $T = \sum_{i=1}^{k} \sum_{j=1}^{n} Y_{ij}$

THEN

$$SST = \sum\sum Y_{ij}^2 - \frac{T^2}{nk}$$

$$SSE = \sum\sum Y_{ij}^2 - \sum_{i=1}^{k} T_i^2 / n$$

$$SS(T_r) = \sum_{i=1}^{k} T_i^2 / n - \frac{T^2}{nk}$$

SEE EXAMPLE ON Pg. 269

FOR UNEQUAL SAMPLE SIZES

$Y_i = \sum_j Y_{ij} / n_i$; $\bar{Y} = \sum_i \sum_j Y_{ij} / \sum_i n_i$; $T_i = \sum_{j=1}^{n_i} Y_{ij}$

$$SST = \sum_i \sum_j (Y_{ij} - \bar{Y})^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} Y_{ij}^2 - \frac{T^2}{\sum_{i=1}^{k} n_i}$$

$$SS(T_r) = \sum_{i=1}^{k} n_i (Y_{ij} - \bar{Y})^2 = \sum_{i=1}^{k} T_i^2 / n_i - \frac{T^2}{\sum_{i=1}^{k} n_i}$$

$$SSE = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y}_i)^2 = \sum_i \sum_j Y_{ij}^2 - \sum_{i=1}^{k} \frac{T_i^2}{n_i}$$

Pg 74

3) FROM TABLE ON Pg. 398

a) $P[X < 2] = F(2) = 0.9772$

b) $P[X < -1.96] = 1 - P[X < 1.96]$
$= 1 - 0.9750$
$= 0.0250$

c) $P[X > 2.58] = 1 - P[X < 2.58]$
$= 1 - 0.9951$
$= 0.0049$

d) $P[X > -2.33] = P[X < 2.33]$
$= 0.9901$

e) $P[0 < X < 1.0] = P[X < 1.0] - P[X < 0]$
$= 0.8413 - 0.5000$
$= 0.3413$

f) $P[0.58 < X < 2.12] = P[2.12] - P[0.58]$
$= 0.9830 - 0.7190$
$= 0.2640$

g) $P[-1.65 < X < -0.84] = P[0.84 < X < 1.65]$
$= P[X < 1.65] - P[X < 0.84]$
$= 0.9505 - 0.7995$
$= 0.1510$

h) $P[-2.42 < X < 1.86] = P[X < 1.86] - P[X < -2.42]$
$= P[X < 1.86] - \{1 - P[X < 2.42]\}$
$= 0.9686 - 1 + 0.9922$
$= 0.9608$

Pg. 74

4) FROM Pg 398

a) $P[X < z] = 0.9265 \Rightarrow z = 1.45$

b) $P[X > z] = 1 - P[X < z] = 0.2843$

$\Rightarrow P[X < z] = 0.7157 \Rightarrow z = 0.57$

c) $P[X > z] = 0.8531 = P[X < -z]$

$\Rightarrow -z = 1.05$

$z = -1.05$

d) $P[-z < X < z] = P[X < z] - P[X < -z]$

$= 2P[0 < X < z] = 0.9312$

$\Rightarrow P[X < z] - P[X < 0] = 0.4656$

$P[X < z] = 0.9656$

$\therefore z = 1.82$

Pg. 75

6) $\mu = 100 \; ; \sigma = 20$

a) $P[x < 97.3] = P\left[z < \frac{97.3 - 100}{20}\right]$

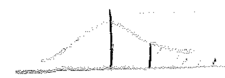
$= P\left[z < \frac{2.7}{20}\right]$

$= P[z < .135]$

$\simeq P[z < 0.13]$

$= 0.5517$

b) $P[x > 110] = P\left[z > \frac{10}{20}\right]$



$= P[z > 0.5]$

$= 1 - P[z < 0.5]$

$= 1 - 0.6915$

$= 0.3085$

c) $P[112.1 < x < 115.8] = P\left[\frac{112.1 - 100}{20} < z < \frac{115.8 - 100}{20}\right]$

$= P\left[\frac{12.1}{20} < z < \frac{15.8}{20}\right]$

$= P[0.605 < z < .79]$

$\simeq P[0.60 < z < 0.79]$

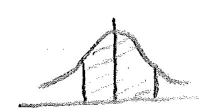
$= P[z < 0.79] - P[z < 0.60]$

$= 0.7852 - 0.7257$

$= 0.0595$

d) $P[95.6 < x < 104.4] = P\left[\frac{95.6 - 100}{20} < z < \frac{104.4 - 100}{20}\right]$

$= P\left[\frac{-4.4}{20} < z < \frac{4.4}{20}\right]$



$= P[-0.22 < z < 0.22]$

$= 2\left[P[z < 0.22] - 0.5\right]$

$= 2[0.5871 - 0.5]$

$= 2[0.0871]$

$= 0.1742$

e) $P[81.3 < x < 92.2] = P\left[\frac{81.3-100}{20} < z < \frac{92.2-100}{20}\right]$

$= P\left[\frac{-18.7}{20} < z < \frac{-7.2}{20}\right]$

$= P[-0.935 < z < -0.36]$

$= P[0.36 < z < 0.935]$

$= P[z < 0.935] - P[z < 0.36]$

$\approx P[z < 0.94] - P[z < 0.36]$

$= 0.8264 - 0.6406$

$= 0.1858$

f) $P[x < 87.3; x > 108.5] = 1 - P[87.3 < x < 108.5]$

$= 1 - P\left[\frac{87.3-100}{20} < z < \frac{108.5-100}{20}\right]$

$= 1 - P\left[\frac{-12.7}{20} < z < \frac{8.5}{20}\right]$

$= 1 - P[-0.635 < z < 0.425]$

$\approx 1 - P[-0.640 < z < 0.430]$

$= 1 - \{P[z < 0.430] - P[z < -0.640]\}$

$= 1 + P[z < -0.640] - P[z < 0.43]$

$= 1 + [1 - P[z < 0.64]] - P[z < 0.43]$

$= 2 - P[z < 0.64] - P[z < 0.43]$

$= 2 - [0.7389 + 0.6664]$

$= 2 - 1.4053$

$= 0.5947$

Pg 75

7) $\mu = 12.8$; $\sigma^2 = 6.25 \Rightarrow \sigma = 2.5$

a) $P[X > 17] = P\left[Z > \frac{17-12.8}{2.5}\right]$

$= P[Z > 1.68]$

$= 1 - P[Z < 1.68]$

$= 1 - 0.9535$

$= 0.0465$

b) $P[X < 11.3] = P\left[Z < \frac{11.3-12.8}{2.5}\right]$

$= P[Z < -0.60]$

$= 1 - P[Z < 0.60]$

$= 1 - 0.7257$

$= .2743$

c) $P[10.1 < X < 14.9] = P\left[\frac{10.1-12.8}{2.5} < Z < \frac{14.9-12.8}{2.5}\right]$

$= P[-1.08 < Z < 0.84]$

$= P[Z < 0.84] - P[Z < -1.08]$

$= P[Z < 0.80] - \{1 - P[Z < 1.08]\}$

$= P[Z < 0.80] + P[Z < 1.08] - 1$

$= P[Z < 0.80] + 0.8599 - 1$

$= 0.7881 + 0.8599 - 1$

$= 0.6480$

d) $P[X < 9.2, X > 15.7] = 1 - P[9.2 < X < 15.7]$

$= 1 - P\left[\frac{9.2-12.8}{2.5} < Z < \frac{15.7-12.8}{2.5}\right]$

$= 1 - P[-1.44 < Z < 1.16]$

$= 1 - P[Z < 1.66] - P[Z < -1.44$

$= 1 - \{P[Z < 1.66] + P[Z < -1.44]$

$= 1 - P[Z < 1.66] + 1 - P[Z < 1.44$

$= 1 - P[Z < 1.66]$

$= 2 - [0.9515 + 0.9251]$

$= 0.8766$

Pg 75

8) $n = 20$; $p = 0.40$

a) $P(x = 10)$

$\mu = np = (20)(0.40) = 8$

$\sigma^2 = np(1-p) = (8)(0.6) = 4.8 \Rightarrow \sigma = 2.2$

$P[X_B = 10] = P[9.5 < X_N < 10.5]$

$= P\left[\frac{9.5 - 8}{2.2} < Z < \frac{10.5 - 8}{2.2}\right]$

$= P\left[\frac{1.5}{2.2} < Z < \frac{2.5}{2.2}\right]$

$= P[0.68 < Z < 1.44]$

$= P[Z < 1.44] - P[Z < 0.68]$

$= 0.9251 - 0.7517$

$= 0.1734$

$P[X_B = 10] = P[X_B \leq 11] - P[X_B \leq 9]$

$= 0.9435 - 0.7553$

$= 0.1882$

b) $P[X_B \leq 12] = P[X_n \leq 12.5]$

$= P\left[Z \leq \frac{12.5 - 8}{2.2}\right]$

$= P\left[Z \leq \frac{4.5}{2.2}\right]$

$= P[Z < 2.0]$

$= 0.9772$

$P[X_B \leq 12] = 0.9790$

Pg 75

9) $n = 400$; $p = 0.5$

$\mu = pn = 200$

$\sigma^2 = pn(1-n) = 100 \Rightarrow \sigma = 10$

a) $P[x_B > 220] = P[x_B \geq 221]$

$\simeq P[X_N > 220.5]$

$= P\left[Z > \frac{220.5 - 200}{10}\right]$

$= P\left[Z > \frac{20.5}{10}\right]$

$= P[Z > 2.05]$

$= 1 - P[Z < 2.05]$

$= 1 - 0.9798$

$= .0202$ (CHECK)

b) $P[185 \leq X_B \leq 215] \simeq P[184.5 < X_N < 215.5]$

$= P\left[\frac{184.5 - 200}{10} < X_N < \frac{215.5 - 200}{10}\right]$

$= P[-1.55 < X_N < 1.55]$

$= 2\{P[X_N < 1.55] - 0.5]$

$= 2\{0.9394 - 0.5\}$

$= 2\{0.4394]$

$= 0.8788$ (CHECK)

10) $n = 600$

$P[x=6] = (\frac{1}{6})(600) = 100$

LET X = # OF 6'S $\Rightarrow p = 1/6$; $n = 600$

$\mu = np = 100$

$\sigma^2 = np(1-p) = (100)\frac{5}{6} = \frac{500}{6} = 83.3 \Rightarrow \sigma = 9.13$

$P[85 \le X_B \le 115] = P[84.5 < X_N < 115.5]$

$= P\left[\frac{84.5-100}{9.13} < Z < \frac{115.5-100}{9.13}\right]$

$= P\left[\frac{-15.5}{9.13} < Z < \frac{15.5}{9.13}\right]$

$= P[-1.26 < Z < 1.26]$

$= 2\,[P[Z < 1.26] - 0.5]$

$= 2\,[0.8962 - 0.5]$

$= 2\,[0.3962]$

$= 0.7924$

$P[X_B < 85, X_B > 115] = 1 - 0.7924$

$= .2076$

Pg. 75

11) $p = \frac{1}{4} = 0.25$

$n = 100$

$\mu = 25$

$\sigma^2 = 25\left(\frac{3}{4}\right) = \frac{75}{4} \Rightarrow \sigma = 4.35$

$P[X_B < 30] \cong P[X_n < 29.5]$

$= P\left[Z < \frac{29.5 - 25}{4.35}\right]$

$= P[Z < 1.03]$

$= 0.8485$

Pg 75

12) $\mu = 0.251$

$\sigma = 0.003$

$$P[0.245 < x < 0.255] = P\left[\frac{0.245-0.251}{0.003} < z < \frac{0.255-0.251}{0.003}\right]$$

$$= P\left[\frac{-.006}{.003} < z < \frac{.004}{.003}\right]$$

$$= P[-2 < z < 1.33]$$

$$= P[z < 1.33] - P[z < -2]$$

$$= P[z < 1.33] - \{1 - P[z < 2]\}$$

$$= P[z < 1.33] + P[z < 2] - 1$$

$$= 0.9082 + 0.9772 - 1$$

$$= 0.8854$$

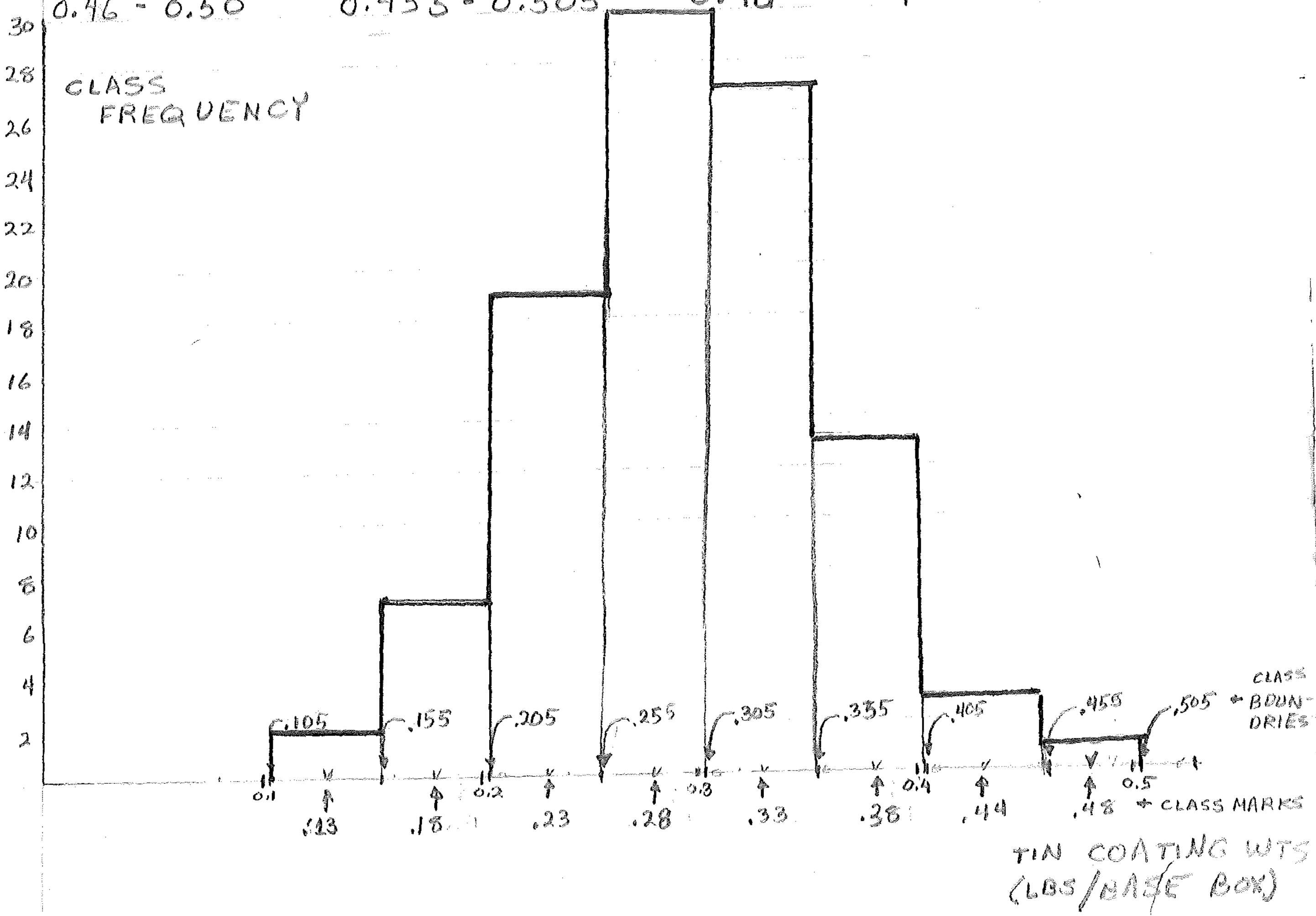
BOB MARKS

9/4

Pg 114

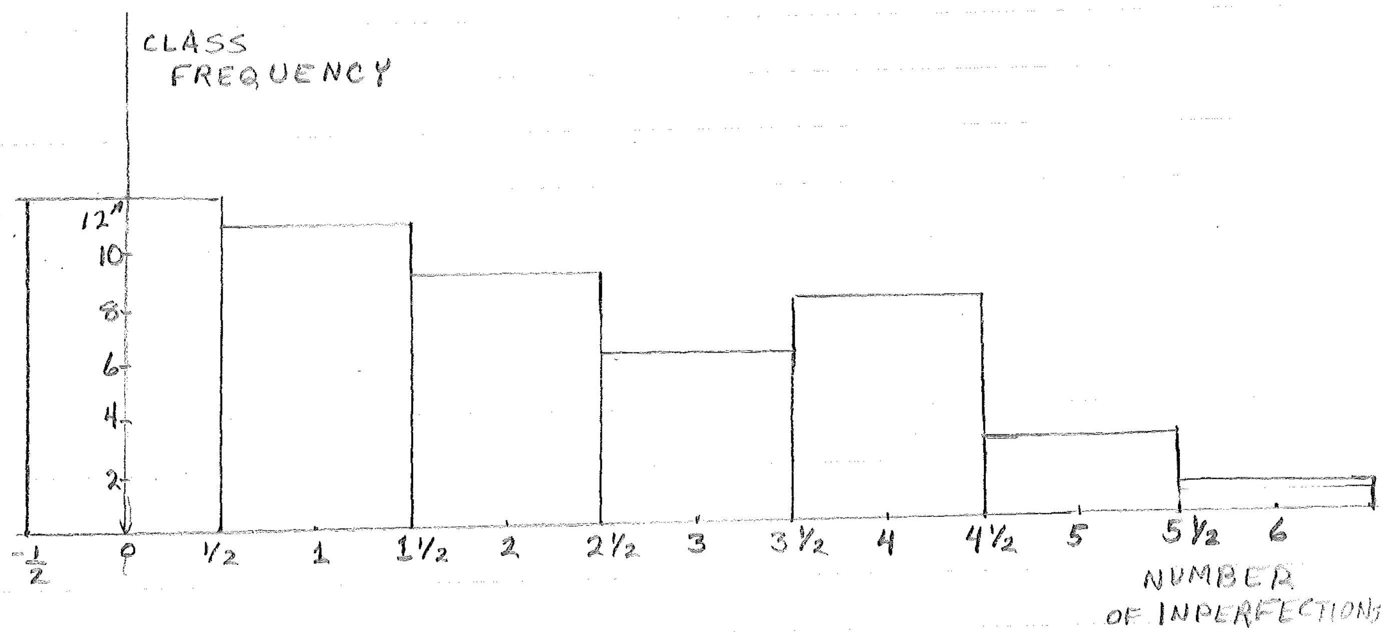
3) CLASS INTERVAL = 0.05 ; RANGE = 0.48 - 0.12 = 0.36.

CLASS LIMITS	CLASS BOUNDRIES	CLASS MARKS	TALLY	FREQ
0.11 - 0.15	0.105 - 0.155	0.13	II	2
0.16 - 0.20	0.155 - 0.205	0.18	IIII II	7
0.21 - 0.25	0.205 - 0.255	0.23	IIII IIII IIII IIII	19
0.26 - 0.30	0.255 - 0.305	0.28	IIII IIII IIII IIII IIII IIII	30
0.31 - 0.35	0.305 - 0.355	0.33	IIII IIII IIII IIII IIII II	27
0.36 - 0.40	0.355 - 0.405	0.38	IIII IIII I	11
0.41 - 0.45	0.405 - 0.455	0.43	III	3
0.46 - 0.50	0.455 - 0.505	0.48	I	1



13a)

OBSERVATION	TALLY	FREQUENCY
0	卌 卌 II	12
1	卌 卌 I	11
2	卌 IIII	9
3	卌 I	6
4	卌 III	8
5	III	3
6	I	1



Pg 123

1) $\sum_{i=1}^{n}(x_i - \bar{x}) = \sum_{i=1}^{n}\left(x_i - \sum_{j=1}^{n} x_j\right)$

$= \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} x_j$

$= 0$

2)

x	x^2
12	144
17	289
24	576
15	225
16	256
8	64
13	169
21	441
30	900
14	196
170	3260

8 - 12 - 13 - 14 - 15 - 16 - 17 - 21 - 24 - 30

MEAN = 15.5

$\bar{x} = \frac{\Sigma x_i}{n} = \frac{170}{10} = 17$

$s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n-1} = \frac{\Sigma x_i^2 - n\bar{x}^2}{n-1} = \frac{3260 - 2890}{9} = \frac{370}{9} = 41.1$

$s = 6.41$

Pg 123

5) $x_i = c u_i + a$

SHOW $\bar{x} = c \cdot \bar{u} + a$; $s_x = c \cdot s_u$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{\sum (c u_i + a)}{n} = \frac{c \sum u_i + na}{n} = c\bar{u} + a$$

$$s_x^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{\sum (c \cdot u_i + a - c\bar{u} + a)^2}{n}$$

$$= \frac{\sum (c \cdot u_i - c\bar{u})^2}{n-1}$$

$$= \frac{\sum (c \cdot u_i)^2 - n(c\bar{u})^2}{n-1}$$

$$= \frac{c^2 \sum u_i^2 - n c^2 \bar{u}^2}{n-1}$$

$$= \frac{c^2 \sum u_i^2 - c^2 \bar{u}^2}{n-1}$$

$$= c^2 \left[\frac{\sum u_i - \bar{u}^2}{n-1} \right]$$

$$= c^2 s_u^2$$

$$s_x = c\, s_u$$

Pg. 135

7)

x	$p(x)$
1	1/4
2	1/4
3	1/4
4	1/4

$\mu = 5/2$; $\sigma^2 = 5/4$

WITHOUT REPLACEMENT

SAMPLE	P(SAMPLE)	$\bar{X}$	$\bar{x}$	$P_{\bar{x}}(\bar{x})$
{1, 2}	1/6	3/2	3/2	1/6
{1, 3}	1/6	4/2	4/2	1/6
{1, 4}	1/6	5/2	5/2	2/6
{2, 3}	1/6	5/2	6/2	1/6
{2, 4}	1/6	6/2	7/2	1/6
{3, 4}	1/6	7/2		

$$\mu_{\bar{x}} = \Sigma \bar{X} \, P_{\bar{x}}(\bar{X})$$
$$= \frac{3}{2}\cdot\frac{1}{6} + \frac{4}{2}\cdot\frac{1}{6} + \frac{5}{2}\cdot\frac{2}{6} + \frac{6}{2}\cdot\frac{1}{6} + \frac{7}{2}\cdot\frac{1}{6}$$
$$= \frac{3+4+10+6+7}{12} = \frac{30}{12} = \frac{5}{2}$$

$$E[\bar{x}^2] = \frac{9}{4}\cdot\frac{1}{6} + \frac{16}{4}\cdot\frac{1}{6} + \frac{25}{4}\cdot\frac{2}{6} + \frac{36}{4}\cdot\frac{1}{6} + \frac{49}{4}\cdot\frac{1}{6}$$
$$= \frac{9+16+50+36+49}{24} = \frac{160}{24} = \frac{80}{12} = \frac{40}{6} = \frac{20}{3}$$

$$\sigma_{\bar{x}}^2 = E[\bar{x}^2] - \mu_{\bar{x}}^2$$
$$= \frac{20}{3} - \frac{25}{4}$$
$$= \frac{80-75}{12} = \frac{5}{12}$$

PLUG: $\sigma_{\bar{x}}^2 = \frac{N-n}{N-1}\,\frac{\sigma^2}{n} = \frac{4-2}{4-1}\,\frac{5/4}{2}$
$$= \frac{2}{3}\,\frac{5}{8}$$
$$= \frac{5}{12}$$

$\mu_{\bar{x}} = \mu_x = 5/2$

SAMPLING WITH REPLACEMENT

SAMPLES	P[SAMPLE]	$\bar{X}$	$\bar{X}$	$P_{\bar{X}}(\bar{x})$
$\{1,1\}$	$\frac{1}{16}$	$2/2$	$2/2$	$1/16$
$\{1,2\}$	$\frac{2}{16}$	$3/2$	$3/2$	$2/16$
$\{1,3\}$	$\frac{2}{16}$	$4/2$	$4/2$	$3/16$
$\{1,4\}$	$\frac{2}{16}$	$5/2$	$5/2$	$4/16$
$\{2,2\}$	$\frac{1}{16}$	$4/2$	$6/2$	$3/16$
$\{2,3\}$	$\frac{2}{16}$	$5/2$	$7/2$	$2/16$
$\{2,4\}$	$\frac{2}{16}$	$6/2$	$8/2$	$1/16$
$\{3,3\}$	$\frac{1}{16}$	$6/2$		
$\{3,4\}$	$\frac{2}{16}$	$7/2$		
$\{4,4\}$	$\frac{1}{16}$	$8/2$		

$\mu_{\bar{x}} = 5/2$ (INSPECTION FROM SYMMETRY)

$$E[\bar{X}^2] = \frac{4}{4}\cdot\frac{1}{16} + \frac{9}{4}\frac{2}{16} + \frac{16}{4}\frac{3}{16} + \frac{25}{4}\cdot\frac{4}{16} + \frac{36}{4}\cdot\frac{3}{16} + \frac{49}{4}\cdot\frac{2}{16} + \frac{64}{4}\cdot\frac{1}{16}$$
$$= \frac{1}{64}[4 + 18 + 48 + 100 + 108 + 98 + 64]$$
$$= \frac{440}{64} = \frac{220}{32} = \frac{110}{16} = \frac{55}{8}$$

$$\sigma_{\bar{x}}^2 = E[\bar{X}^2] - \mu_{\bar{x}}^2$$
$$= \frac{55}{8} - \frac{25}{4}$$
$$= \frac{5}{8}$$

PLUG:

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} = \frac{5/4}{2} = 5/8 \qquad \text{(CHECK)}$$
$$\mu_{\bar{x}} = \mu_x = 5/2$$

Pg 136

10) $n = 64$; $\mu = 112$; $\sigma^2 = 144$

FIND $P[\bar{x} > 114.5]$

$\mu_{\bar{x}} = \mu = 112$

$\sigma_{\bar{x}}^2 = \sigma^2/n = \frac{144}{64} = \frac{72}{32} = \frac{36}{16} = \frac{18}{8} = \frac{9}{4} \Rightarrow \sigma_{\bar{x}} = 3/2$

THEN: $P[\bar{x} > 114.5] = P\left[z > \frac{114.5 - 112}{3/2}\right]$

$= P\left[z > \frac{2.5}{3/2}\right]$

$= P\left[z > \frac{5 \cdot 2}{2 \cdot 3}\right]$

$= P[z > 1.667]$

$= 1 - P[z < 1.667]$

$= 1 - 0.9525$

$= .0475$

11) $n = 100$

$\mu = 53$

$\sigma^2 = 400$

FIND $P[50 < \bar{x} < 56]$

$\mu_{\bar{x}} = \mu_x = 53$

$\sigma_{\bar{x}}^2 = \sigma_x^2 / n = \frac{400}{100} = 4 \Rightarrow \sigma_{\bar{x}} = 2$

$$\Rightarrow P[50 < \bar{x} < 56] = P\left[\frac{50-53}{2} < z < \frac{56-53}{2}\right]$$

$$= P[-1.5 < z < 1.5]$$

$$= 2\left[P[z < 1.5] - 0.5\right]$$

$$= 2[0.9332 - 0.5]$$

$$= 2[0.4332]$$

$$= 0.8664$$

12) $\sigma = 1.5$

$n = 4$

$\mu = 10$

FIND $P[\bar{x} < 7.75]$

$\mu_{\bar{x}} = \mu = 10$

$\sigma_{\bar{x}} = \sigma/\sqrt{n} = \frac{1.5}{2} = 0.75$

$$P[\bar{x} < 7.75] = P\left[z < \frac{7.75 - 10}{0.75}\right]$$
$$= P\left[z < -\frac{2.25}{0.75}\right]$$
$$= P[z < -3]$$
$$= 1 - P[z < 3]$$
$$= 1 - 0.9987$$
$$= .0013$$

YOU'RE PRETTY SAFE

Pg 136

13) $\sigma = 0.05$

$n = 16 \Rightarrow \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.05}{4} = 0.0125$

FIND $P[|\bar{X} - \mu| < 0.02]$

$\mu = \mu_{\bar{x}}$

$$\Rightarrow P[|\bar{X} - \mu| < 0.02] = P[|\bar{X} - \mu_{\bar{x}}| < 0.02]$$
$$= P[|Z| < \frac{0.02}{0.0125}]$$
$$= P[|Z| < 1.6]$$
$$= P[-1.6 < Z < 1.6]$$
$$= 2\{P[Z < 1.6] - 0.5\}$$
$$= 2(0.9452 - 0.5)$$
$$= 2(0.4450)$$
$$= 0.8900$$

Pg 141

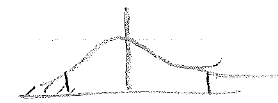
1) $n = 16$; $\bar{x} = 48$; $s = 5.2$

FIND $P[\mu > 52] = P\left[\frac{\mu - \bar{x}}{s/\sqrt{n}} > \frac{52 - \bar{x}}{s/\sqrt{n}}\right]$

$= P\left[\frac{\bar{x} - \mu}{s/\sqrt{n}} < \frac{48 - 52}{5.2/\sqrt{16}}\right]$

$= P[t_\gamma < -3.08]$

$= P[t_\gamma > 3.08]$



NOW $P[t_\gamma > 3.08] < 0.005$

$\Rightarrow P[\mu > 52] < 0.005$

NOT TO HOT A CLAIM

Pg 141

2) $x_i = (28, 15, 19, 30, 23) \Rightarrow n = 5$

$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = 23$

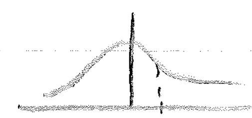
$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i - \bar{x}^2\right] = 38.75 \Rightarrow s = 6.225$

FIND $P[\mu < 20] = P\left[\frac{\mu - \bar{x}}{s/\sqrt{n}} < \frac{20-23}{6.225/\sqrt{5}}\right]$

$= P\left[\frac{\mu - \bar{x}}{s/\sqrt{n}} < -0.964\right]$

$= P\left[\frac{\bar{x} - \mu}{s/\sqrt{n}} > 0.964\right]$

$= P[t_4 > t_{4;\alpha} = 0.964]$



$> .10$

NOT SO GOOD A CLAIM, SINCE $P[\mu < 20] < 50\%$

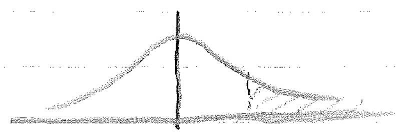
Pg 141·2

3) $n = 10;\ \bar{X} = 2.53;\ s = 0.2$

FIND $P[\mu \simeq 2.5] = P\left[\frac{\mu - \bar{x}}{s/\sqrt{n}} \simeq \frac{2.5 - 2.53}{0.2/\sqrt{9}}\right]$

$= P\left[\frac{\bar{x} - \mu}{s/\sqrt{n}} \simeq 4.74\right]$

$= P\left[t_r \simeq t_{r;9} = 4.74\right]$



OUT OF CONTROL

Pg 142

4) $n = 17$; $\gamma = 16$; $\sigma^2 = 64$

FIND $P[s^2 < 115.38]$

$$= P\left[\frac{(n-1)s^2}{\sigma^2} < \frac{(n-1)115.4}{\sigma^2}\right]$$

$$= P[\chi^2_\gamma < 28.84 = \chi^2_{\gamma;a}]$$

$$= P[\chi^2_{16} < 28.64] \sim 0.025$$

so $P[s^2 < 115.38] \sim 0.025$

$\Rightarrow P[\text{CLAIM WILL BE REVECTED}]$

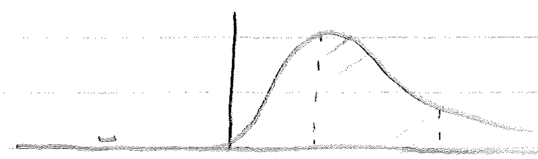
$$= P[s^2 < 115.38] \sim 0.025$$

Pg 142

5) $n = 27$; $\nu = 26$; $\sigma^2 = 16.8$

FIND $P[3 < S^2 < 5.2] = P\left[\frac{3(n-1)}{\sigma^2} < \frac{S^2(n-1)}{\sigma^2} < \frac{5.2(n-1)}{\sigma^2}\right]$

$= P[13.93 < \chi^2_{26} < 42.2]$



$= P[\chi^2_{26} > 13.93] - P[\chi^2_{26} > 42.2]$

$\cong 0.975 - 0.025$

$= 0.950$

Pg 142

6) $n_1 = 21$; $n_2 = 20$

$r_1 = 20$; $r_2 = 19$

FIND $P\left[\left(\frac{s_1}{s_2}\right)^2 > 3\right] = P\left[\left(\frac{s_1}{s_2}\right)^2 > 3\right]$

$= P\left[F_{r_1 r_2;\alpha} > 3\right]$

$= P\left[F_{20,19;\alpha} > 3\right] = 0.01$

7) $P\left[\left(\frac{s_1}{s_2}\right)^2 > 3\right] + P\left[\left(\frac{s_2}{s_1}\right)^2 > 3\right] = 0.01 + 0.01 = 0.02$

Pg 149

1-2) $n = 50$

$\bar{X} = 0.2210$

$S = 0.024$

$\alpha = 0.05$

CONFIDENCE INTERVAL FOR MEAN DEFINED FROM:

$$\bar{X} - t_{n-1;\alpha/2} S/\sqrt{n} \leq \mu \leq \bar{X} + t_{n-1;\alpha/2} S/\sqrt{n}$$

$t_{n-1;\alpha/2} \simeq Z_{\alpha/2} = Z_{0.025} = 1.960$

$t_{n-1;\alpha/2} S/\sqrt{n} = (1.96)(0.024)/\sqrt{50}$

$= .0067$

$\bar{X} \pm t_{n-1;\alpha/2} S/\sqrt{n} = 0.2210 \pm .0067$

SO ONE MAY SAY WITH 95% CERTAINTY THAT THE TRUE MEDIAN LIES IN THE CONFIDENCE INTERVAL

$$0.2143 \leq \mu \leq 0.2277$$

Pg 149

3) $n = 40$

$\bar{x} = 42.5$

$s = 3.8$

a) $1-\alpha = 0.99 = P\left[\bar{x} - t_{n-1;\alpha/2}\, s/\sqrt{n} < \mu < \bar{x} + t_{n-1;\alpha/2}\, s/\sqrt{n}\right]$

$t_{n-1;\alpha/2} \simeq z_{\alpha/2} = z_{0.005} = 2.576$

$t_{n-1;\alpha/2}\, s/\sqrt{n} = (2.576)(3.8)/\sqrt{40} = 1.55$

∴ ERROR WILL NOT EXCEED 1.55 MINUTES

b) CONFIDENCE INTERVAL;

$\bar{x} - t_{n-1;\alpha/2}\, s/\sqrt{n} < \mu < \bar{x} + t_{n-1;\alpha/2}\, s/\sqrt{n}$

$42.5 - 1.55 < \mu < 42.5 + 1.55$

$(41.95 < \mu < 43.05)$ MINUTES

FOR $\alpha = 0.01$

FOR $\alpha = 0.02$

$t_{n-1;\alpha/2} \simeq z_{\alpha/2} = z_{0.01} = 2.326$

$t_{n-1;\alpha/2}\, s/\sqrt{n} = (2.326)(3.8)/\sqrt{40} = 1.4$

CONFIDENCE INTERVAL:

$\bar{x} - t_{n-1;\alpha/2}\, s/\sqrt{n} < \mu < \bar{x} + t_{n-1;\alpha/2}\, s/\sqrt{n}$

$42.5 - 1.4 < \mu < 42.5 + 1.4$

$41.1 < \mu < 43.9$

Pg 150

4) THE DESIRED CONFIDENCE INTERVAL IS:

$$\bar{X} - 1 < \mu < \bar{X} + 1$$

ERGO: $t_{n-1;\alpha/2}\ S/\sqrt{n} = 1$

OR $t_{n-1;\alpha/2} = \sqrt{n}/S$

SINCE $n = 40$; $t_{n-1;\alpha/2} \simeq Z_{\alpha/2}$

SO $Z_{\alpha/2} = \sqrt{n}/S$

$= \sqrt{40}/3.8$

$= 1.67$

$\therefore\ t_{n-1;\alpha/2} = t_{39;\alpha/2}$

$= t_{\infty;\alpha/2}$

$= 1.67$

$t_{\infty;0.05} = 1.645$

$\Rightarrow \alpha/2 \simeq 0.05$

OR $\alpha \simeq 0.1$

SO THERE IS A 90% CHANCE THE TRUE TIME LIES IN THE CONFIDENCE INTERVAL $41.5 \le \mu < 43.5$

Pg 150

5) $\alpha = 0.05$; $\sigma = 16$

$Z_{\alpha/2} \, \sigma/\sqrt{n} = 2$

OR $n = \left[\frac{1}{2} Z_{\alpha/2} \, \sigma\right]^2$

$= \left[\frac{1}{2} Z_{0.025} \, (16)\right]^2$

$= [(8)(\quad)]^2$

$= 22.9$

$\Rightarrow n = 23$

Pg 150

7) $n = 25$

$\bar{x} = 12.81$

$s = 0.04$

$\alpha = 0.05$

$t_{n-1;\alpha/2} = t_{24;0.025} = 2.064$

$t_{n-1;\alpha/2}\, s/\sqrt{n} = (2.064)(0.04)/5$

$= 0.017$

CONFIDENCE INTERVAL THEN GIVEN BY:

$12.81 - 0.017 < \mu < 12.81 + 0.017$

$12.79 < \mu < 12.83$

Pg 150

8) $n = 16$

$\bar{X} = 0.338$

$S = 0.012$

a) $\alpha = 0.01$

$t_{n-1;\alpha/2} = t_{15;0.005}$
$= 2.947$

$t_{n-1;\alpha/2}\, S/\sqrt{n} = (2.947)(1.2 \times 10^{-2})/4$
$= 8.85 \times 10^{-3}$
≈ 0.009

CONFIDENCE INTERVAL:

$0.338 - 0.009 < \mu < 0.338 + 0.009$

$0.329 < \mu < 0.347$

b) $\alpha = 0.05$

$t_{n-1;\alpha/2} = t_{15;0.025}$
$= 2.13$

$t_{n-1;\alpha/2}\, S/\sqrt{n} = (2.13)(1.2 \times 10^{-2})/4$
$= 0.006$

CONFIDENCE INTERVAL

$0.338 - 0.006 < \mu < 0.338 + 0.006$

$0.332 < \mu < 0.344$

Pg 150

9) $\alpha = 0.1$

$n = 100$

$\sigma^2 = 1.21 \Rightarrow \sigma = 1.10$

$\bar{x} = 5.68$

$Z_{\alpha/2} = Z_{0.05} = 1.645$

$Z_{\alpha/2}\, \sigma/\sqrt{n} = (1.645)(1.10)/10$

$= 0.180$

CONFIDENCE INTERVAL:

$5.68 - 0.18 < \mu < 5.68 + 0.18$

$5.50 < \mu < 5.86$

Pg 150

10) $\alpha = 0.01$

$n = 20$

$\bar{X} = 6.51 \times 10^3$

$s = 1.15 \times 10^3$

$t_{n-1;\alpha/2} = t_{19;0.005} = 2.86$

$t_{n-1;\alpha/2}\, s/\sqrt{n} = (2.68)(1.15 \times 10^3)/\sqrt{20}$

$= 790$

CONFIDENCE INTERVAL

$6510 - 790 < \mu < 6510 + 790$

$5720 < \mu < 7300$

THERE IS A 99% CHANCE THAT THE TRUE MEAN MILEAGE LIES IN THE INTERVAL

5720 TO 7300 MILES

Pg 150

11) $n = 81$

$\bar{X} = 9.87$

$s = 5.14$

$t_{n-1;\alpha/2} = t_{80;\alpha/2}$

$\approx t_{\infty;\alpha/2}$

$= z_{\alpha/2}$

$t_{n-1;\alpha/2} \, s/\sqrt{n} = 1$

$\Rightarrow z_{\alpha/2} = \frac{\sqrt{n}}{s}$

$= 9/5.14$

$= 1.76$

$\therefore \alpha/2 = 1 - 0.96$

$= 0.04$

$\alpha = 0.08$

STATEMENT HAS A 92% PROBABILITY OF TRUTH

Pg 150

12) $\alpha = 0.10$

$\bar{X} = 9.87$

$S = 5.14$

DUE TO BEHAVIOR OF PREVIOUS PROBLEM, ASSUME $t_{n-1;\alpha/2}$ IS A NORMAL DISTRIBUTION, THEN

$$Z_{\alpha/2}\, S/\sqrt{n} = 0.25$$

OR

$$n = \left[4 Z_{\alpha/2} S\right]^2$$
$$= \left[(4)\, Z_{0.05}\, (5.14)\right]^2$$
$$= \left[(4)(1.645)(5.14)\right]^2$$
$$= 1.14 \times 10^3 \text{ SAMPLES}$$

BOB MARKS

10/10

Pg 160

1) a) $H_0: \mu = 400$: $H_1: \mu < 400$

$n = 36$; $\sigma = 48$; REJECT FOR $\bar{X} < 390$

$\alpha = P[\text{TYPE I ERROR}] = P[\text{REJECTING } H_0 \text{ GIVEN } \mu = 400]$

$= P[\bar{X} < 390 \text{ GIVEN } \mu = 400]$

$= P\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = Z < \frac{390-400}{48/6}\right]$

$= P[Z < \frac{-10}{8} = -1.25]$

$= 1 - P[Z < 1.25]$

$= 1 - 0.8944$

$= 0.1056$

b) $\beta = P[\text{TYPE II ERROR GIVEN } \mu = 380]$

$= P[\text{ACCEPTING } H_0 \text{ WHEN } \mu = 380]$

$= P[\bar{X} > 390 \text{ GIVEN } \mu = 380]$

$= P\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = Z > \frac{390-380}{48/6}\right]$

$= P[Z > 1.25]$

$= 1 - P[Z < 1.25]$

$= 0.1056$

THE TYPE II ERROR IS LESS THAN THAT COMPUTED IN THE TEXT DUE THE SHORTER INDIFFERENCE INTERVAL (STRICTER DECISION RULE)

Pg 160

2) $H_0: \mu = 400 \qquad H_1: \mu \neq 400$

$n = 36; \quad \sigma = 48 \Rightarrow \sigma_{\bar{x}} = \sigma/\sqrt{n} = 8$

REJECT H_0 FOR $415.7 < \bar{x} < 384.3$

$g(410) = P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = 410]$

$= P[384.3 < \bar{x} < 415.7 \text{ GIVEN } \mu = 410]$

$= P\left[\frac{384.3 - 410.0}{8} < \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = z < \frac{415.7 - 410.0}{8}\right]$

$= P\left[-\frac{25.7}{8} < z < \frac{5.7}{8}\right]$

$= P[-3.21 < z < 0.712]$

$= P[z < 0.712] - P[z < -3.21]$

$= P[z < 0.712] + P[z < 3.21] - 1$

$= 0.7611 + 0.9993 - 1$

$= 0.7604$

$g(430) = P[384.3 < \bar{x} < 415.7 \text{ GIVEN } \mu = 430]$

$= P\left[\frac{384.3 - 430}{8} < \frac{\bar{x} - \mu}{\sigma_{\bar{x}}} = z < \frac{415.3 - 430}{8}\right]$

$= P[-5.71 < z < -1.79]$

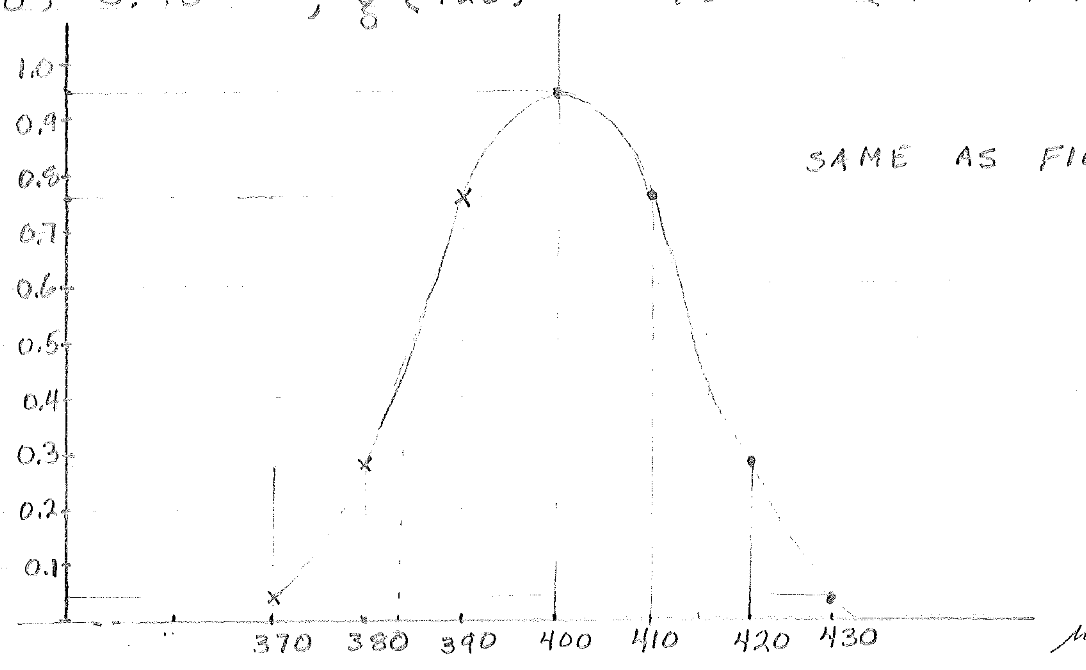
$\simeq P[z < -1.79]$

$= 1 - P[z < 1.79]$

$= 1 - 0.9633$

$= 0.0367$

$g(400) = 0.95 \quad ; \quad g(420) = 0.295 \qquad$ (FROM TEXT)



SAME AS FIG. 8-7

Pg 160-1

3) $H_0: \mu = 0$; $H_1: \mu > 0$

$n = 9$, $\sigma = 1$, $\alpha = 0.05$

a) $P[\text{TYPE I ERROR}] = \alpha$

$\alpha = P[\text{REJECTING } H_0 \text{ GIVEN } \mu = 0]$

$= P[\bar{X} > d_b \text{ GIVEN } \mu = 0]$

$= P[\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} > \frac{d_b - \mu}{\sigma_{\bar{x}}}]$

$= P[Z > \frac{d_b - 0}{1/3} = 3d_b]$

$= P[Z > 3d_b]$

$= 1 - P[Z < 3d_b] = 0.05$

OR $P[Z < 3d_b] = 0.95$

$\Rightarrow 3d_b \approx 1.645$

$\therefore d_b = 0.548$

H_0 WILL BE REJECTED FOR $\bar{X} > 0.548$

b) $\beta = P[\text{TYPE II ERROR}]$

$g(0.5) = P[\text{TYPE II ERROR GIVEN } \mu = 0.5]$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = 0.5]$

$= P[\bar{X} < 0.548] \text{ GIVEN } \mu = 0.5$

$= P[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{0.548 - 0.5}{1/3}]$

$= P[Z < 3 \times 0.048 = 0.144]$

≈ 0.557

$g(1.0) = P[Z < \frac{0.548 - 1.0}{1/3}]$

$= P[Z < -1.36]$

$= 1 - P[Z < 1.36]$

$= 1 - 0.9131$

$= .0869$

$g(1.5) = P[Z < \frac{0.548 - 1.5}{1/3}]$

$= P[Z < -2.86]$

$= 1 - P[Z < 2.86]$

$= 1 - 0.9979$

$= .0021$

Pg 161

4) CASE a ; $n = 25 \Rightarrow \sigma_{\bar{x}} = 1/5$

$\alpha = P[\text{TYPE I ERROR}]$

$= P[\text{REJECTING } H_0; \mu = 0]$

$= P[\bar{X} > d_{ba} \text{ GIVEN } \mu = 0]$

$= P\left[Z = \frac{\bar{X} - \mu}{\sigma_{\bar{x}}} > \frac{d_{ba} - 0}{1/5} = 5 d_{ba}\right]$

$= P[Z > 5 b_{ba}]$

$5 d_{ba} = 1.644 \Rightarrow d_{ba} = 0.3288 \Leftarrow$ (DECISION BOUNDRY)

$g_a(0.5) = P[\text{TYPE II ERROR GIVEN } \mu = 0.5]$

$= P[H_0 \text{ IS ACCEPTED GIVEN } \mu = 0.5]$

$= P[\bar{X} < d_{ba} \text{ GIVEN } \mu = 0.5]$

$= P[\bar{X} < 0.328 \text{ GIVEN } \mu = 0.5]$

$= P\left[Z = \frac{\bar{X} - \mu}{\sigma_{\bar{x}}} < \frac{0.328 - 0.5}{1/5}\right]$

$= P[Z < -0.86]$

$= 1 - P[Z < 0.86]$

$= 1 - 0.805$

$= 0.195$

$g_a(1.0) = P[\bar{X} < 0.328 \text{ GIVEN } \mu = 1.0]$

$= P\left[\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} = Z < \frac{0.328 - 1.0}{1/5}\right]$

$= P[Z < -3.36]$

$= 1 - P[Z < 3.36]$

$= 1 - 0.9996$

$= 0.0004$

$g_a(1.5) = P[\bar{X} < 0.328 \text{ GIVEN } \mu = 1.5]$

$= P\left[\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} = Z < \frac{0.328 - 1.5}{1/5} = -5.86\right] = 0$

$g_a(0) = P\left[\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} = Z < \frac{0.328}{1/5} = 1.64\right]$

$= 0.95$

$g_a(0.25) = P\left[\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} = Z < \frac{0.328 - 0.25}{1/5} = 0.390\right]$

$= 0.652$

CASE b; $n = 100 \Rightarrow \sigma_{\bar{x}} = 1/10$

$\alpha = 0.05 = P[\text{TYPE I ERROR}]$

$= P[H_0 \text{ REJECTED GIVEN } \mu = 0]$

$= P[\bar{X} > d_{bb}]$

$= P[Z > 10 d_{bb}]$

$\Rightarrow 10 d_{bb} = 1.644$

$d_{bb} = 0.1644$

$g(1) = P[\bar{X} < d_{bb} \text{ GIVEN } \mu = 1]$

$= P[Z < \frac{0.1644 - 1}{1/10} = -8.356] \cong 0$

$g(0.25) = P[Z < \frac{0.1644 - 0.25}{1/10} = -0.856]$

$= 1 - P[Z < 0.856]$

$= 1 - 0.801$

$= 0.199 \approx 0.2$

$g(0.5) = P[Z < \frac{0.1644 - 0.50}{1/10} = -3.36]$

$= 1 - P[Z < 3.36]$

$= 1 - 0.9996$

$= 0.0004$

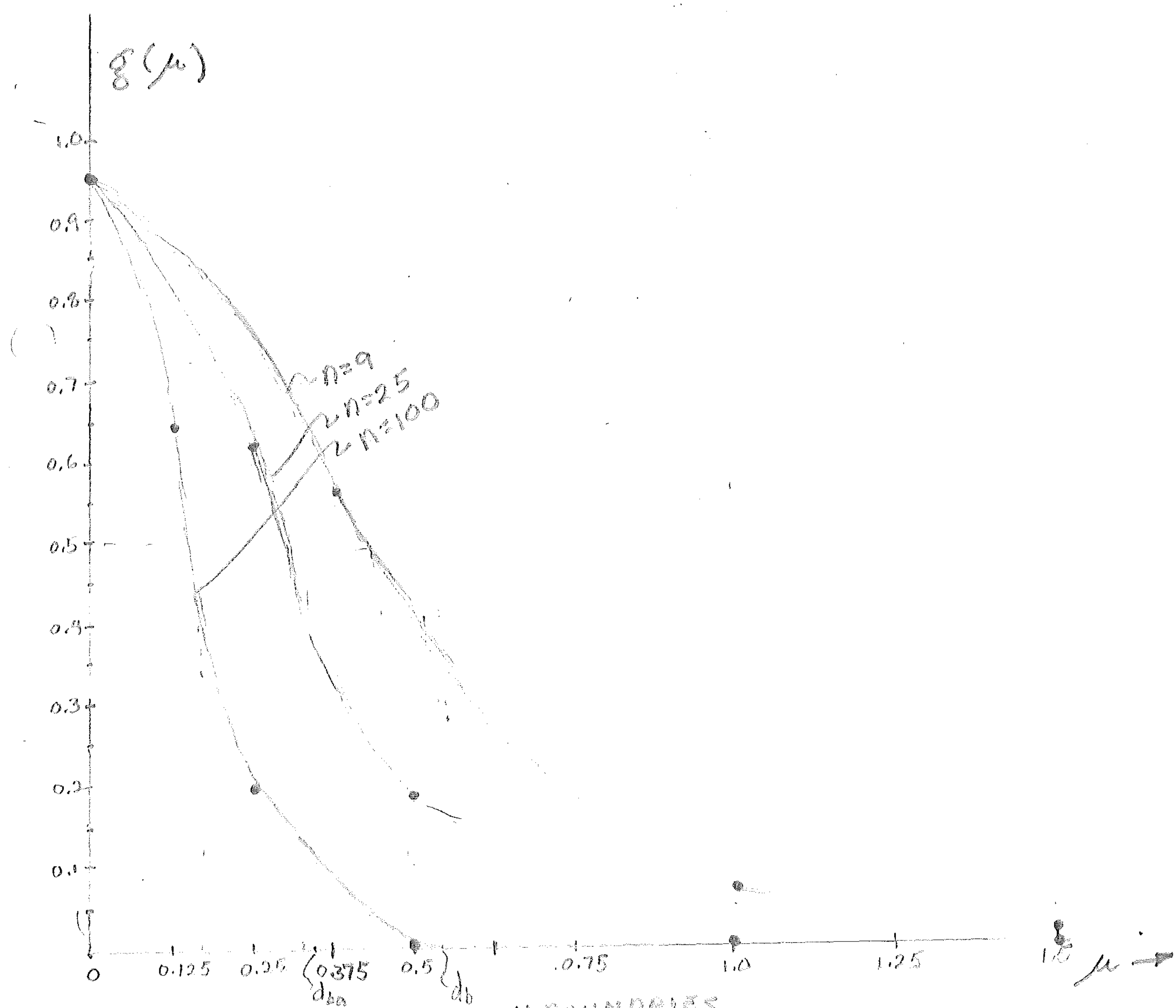
$g(0) = 1 - \alpha = 0.95$

$g(0.125) = P[Z < \frac{0.1644 - 0.125}{1/10}]$

$= P[Z < 0.394]$

$= 0.648$

O.C. CURVES FOR PROBLEMS 3,4, & 5



INFLECTION PTS. @ DECISION BOUNDARIES

Pg 161

5) $H_0: \mu = 0$

$H_1: \mu = 0.3$

$\beta = P[\text{TYPE II ERROR}]$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = 0.3]$

$= P[\bar{X} < d_b]$ GIVEN $\mu = 0.3$

FROM SKETCHED OC CURVE:

$g(0.3$ WITH $n = 9) \simeq 0.68$

$g(0.3$ WITH $n = 25) \simeq 0.4$

$g(0.3)$ WITH $n = 100) \simeq 0.1$

SO IT WOULD TAKE A SAMPLE SIZE OF ABOUT 100.

ANALYTIC CHECK:

$P[\bar{X} < d_b \text{ GIVEN } \mu = 0.3; n = 100] = P\left[Z = \frac{\bar{X} - \mu}{\sigma_{\bar{x}}} < \frac{0.164 - 0.3}{1/10}\right]$

$= P[Z < -1.36] = 1 - P[Z < 1.36]$

$= 1 - 0.913$

$= .087$

SLIGHTLY < 0.1

FOR $n = 25$

$P[\bar{X} < d_b \text{ GIVEN } \mu = 0.3] = P\left[Z < \frac{0.329 - 0.3}{1/5}\right]$

$= P[Z < 0.145]$

$= 0.56$

MUCH TOO LARGE. THE USE OF $n = 9$ WOULD YIELD AN EVEN LARGER VALUE.

$\Rightarrow$ USE $n = 100$

Pg 161

7) $H_0; \mu = 0$

$H_1; \mu > 0$

$n = 100$

$\sigma^2 = 1 \Rightarrow \sigma_{\bar{x}} = 1/10$

REJECT H_0 FOR $\bar{X} > 0.233$

a) $\alpha = P[\text{TYPE I ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu = 0]$

$= P[\bar{X} > 0.233 \text{ GIVEN } \mu = 0]$

$= P[Z > \frac{0.233 - 0}{1/10} = 2.33]$

$= 1 - P[Z < 2.33]$

$= 1 - 0.9901$

$\simeq 0.01$

b) $\beta = P[\text{TYPE II ERROR}]$

$= P[\text{ACCEPTING } H_0]$

$g(0.1) = \beta \text{ GIVEN } \mu = 0.1$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = 0.1]$

$= P[\bar{X} < d_b \text{ GIVEN } \mu = 0.1]$

$= P[\bar{X} < 0.233 \text{ GIVEN } \mu = 0.1]$

$= P[Z < 1.33]$

$= 0.908$

$g(0.2) = P[Z < 0.33]$

$= 0.629$

$g(0.3) = P[Z < 10(0.233 - 0.3)]$

$= P[Z < -0.67]$

$= 1 - P[Z < 0.67]$

$= 1 - 0.749$

$= 0.251$

$g(0.4) = P[Z < 10(0.233 - 0.4)]$

$= P[Z < -1.67] = 1 - P[Z < 1.67]$

$= 1 - 0.9525$

$= 0.0475$

Pg. 161

8) a) REJECT H_0 FOR $\bar{X} > 0.196$; $\sigma_{\bar{X}} = 1/10$

$\alpha = P[\text{TYPE II ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu = 0]$

$= P[\bar{X} > 0.196 \text{ GIVEN } \mu = 0]$

$= P[Z > 1.96]$

$= 1 - 0.9750 \Rightarrow g(0) = 0.975$

$= 0.0250$

$g(0.1) = P[\bar{X} < 0.196 \text{ GIVEN } \mu = 0.1]$

$= P[Z < 0.96]$

$= 0.832$

$g(0.2) = P[\bar{X} < 0.196 \text{ GIVEN } \mu = 0.2]$

$= P[Z < -0.04]$

$= 1 - 0.516$

$= 0.484$

$g(0.3) = P[\bar{X} < 0.196 \text{ GIVEN } \mu = 0.3]$

$= P[Z < -1.04]$

$= 1 - 0.851$

$= 0.149$

$g(0.4) = P[Z < \frac{0.196 - 0.4}{1/10}]$

$= P[Z < -0.2.04]$

$= 1 - 0.979$

$= 0.021$

b) REJECT H_0 FOR $\bar{X} > 0.128$

$\alpha = P[\text{TYPE II ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu]$

$= P[\bar{X} > 0.128 \text{ GIVEN } \mu = 0]$

$= P[Z > 1.28]$

$= 1 - 0.9 \Rightarrow g(0) = 0.9$

$= 0.1$

$g_b(0.1) = P\left[Z < \frac{0.128 - 0.1}{1/10} = 0.28\right]$

$= 0.610$

$g_b(0.2) = P[Z < -0.72]$

$= 1 - 0.764$

$= 0.236$

$g_b(0.3) = P[Z < -1.72]$

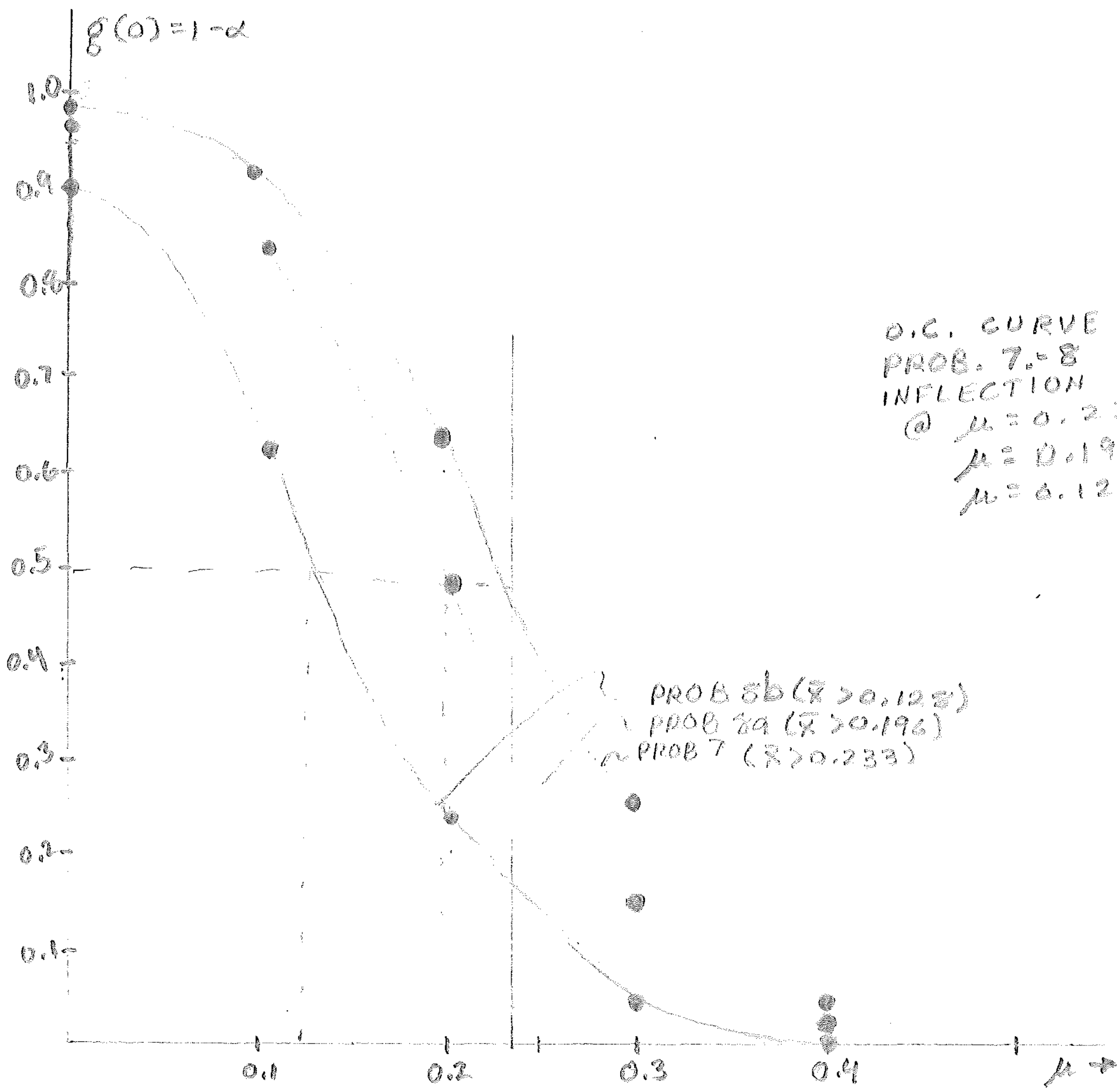
$= 1 - 0.957$

$= .043$

$g_b(0.4) = P[Z < -2.72]$

$= 1 - 0.997$

$= 0.003$



O.C. CURVE FOR
PROB. 7.-8
INFLECTION PTS
@ $\mu = 0.233$ $(\bar{X} > 0.233)$
$\mu = 0.196$ $(\bar{X} > 0.196)$
$\mu = 0.128$ $(\bar{X} > 0.128)$

Pg 161

9) $H_0 : \mu = 0$

$H_1 : \mu \neq 0$

$n = 25$

$\sigma^2 = 1 \Rightarrow \sigma_{\bar{x}} = 1/5$

$\alpha = 0.05 = P[\bar{X} > d \text{ OR } \bar{X} < -d]$ GIVEN $\mu = 0$

$= P[Z > 5d \text{ OR } Z < -5d]$

$= 2[1 - P[Z < 5d]]$

$0.025 = 1 - P[Z < 5d]$

$P[Z < 5d] = 0.975$

$\Rightarrow 5d = 1.96$

OR $d = 0.392$

b) $\beta = P[\text{TYPE II ERROR}]$

$g(\mu) = P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu \neq 0]$

$= P[-d < \bar{X} < d]$ GIVEN $\mu \neq 0$

$g(0) = 1 - \alpha = 0.95$

$g(0.2) = P\left[\frac{-0.392 - 0.2}{1/5} < Z < \frac{0.392 - 0.2}{1/5}\right]$

$= P[-2.96 < Z < 0.960]$

$= P[Z < 0.96] - P[Z < -2.96]$

$= 0.8315 - [1 - P[Z < 2.96]]$

$= 0.8315 - [1 - 0.9985]$

$= 0.8315 - 0.0015$

$= 0.8300$

$g(0.4) = P\left[\frac{-0.392 - 0.4}{1/5} < Z < \frac{0.392 - 0.4}{1/5}\right]$

$= P[-3.96 < Z < -0.04]$

$\simeq P[Z < -0.04]$

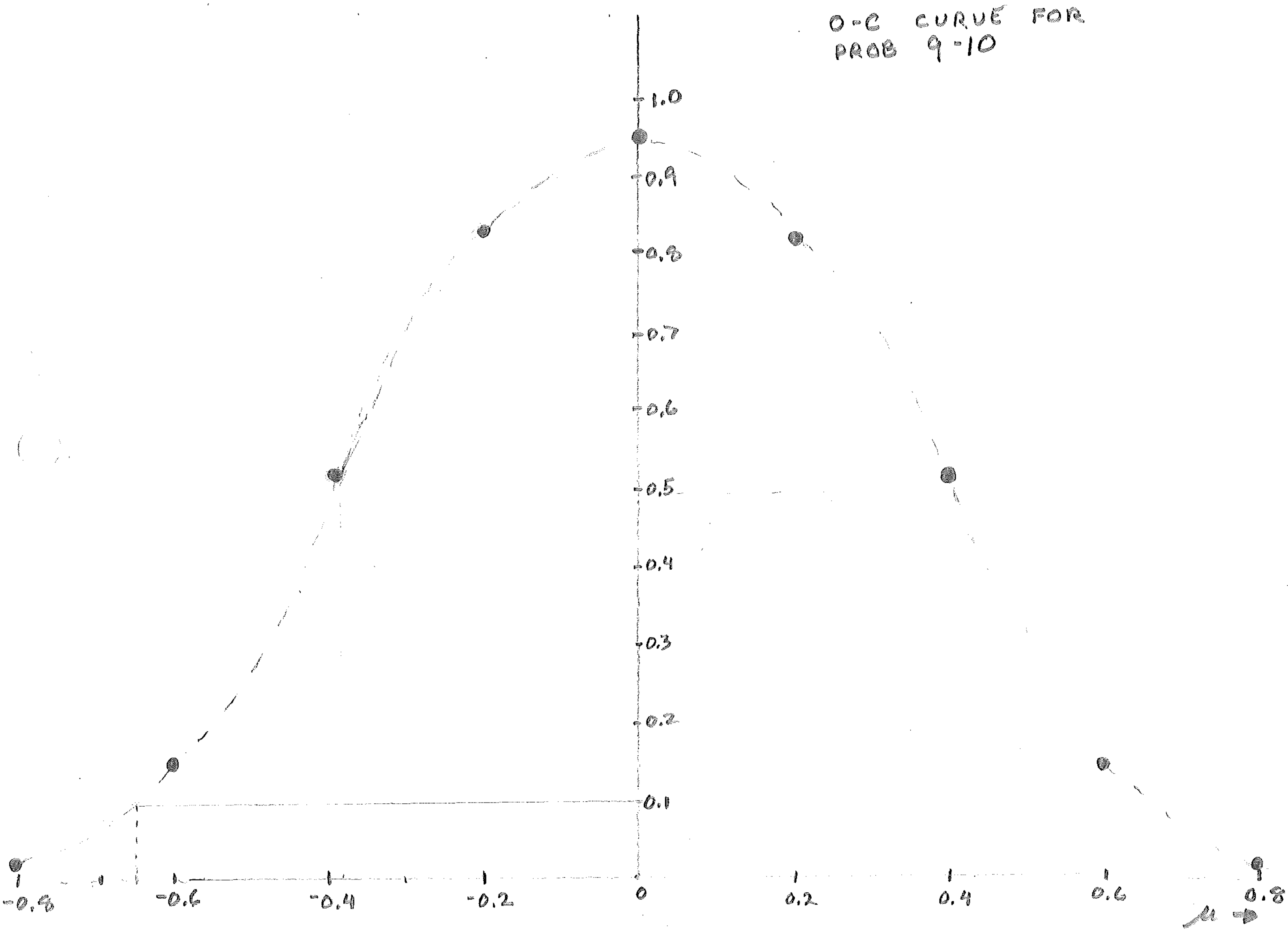
$= 1 - P[Z < 0.04]$

$= 0.516$

$$g(0.6) \doteq P\left[Z < \frac{0.392 - .6}{1/5}\right]$$
$$= P[Z < {}^{-}1.04]$$
$$= 1 - 0.8508$$
$$= 0.1492$$

$$g(0.8) \doteq P\left[Z < \frac{0.392 - 0.8}{1/5}\right]$$
$$= P[Z < {}^{-}2.04]$$
$$= 1 - 0.9793$$
$$= .0207$$

$$g(\mu_n) = g(-\mu_n)$$



O-C CURVE FOR
PROB 9-10
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
μ

Pg 161

10) FROM CURVE:

$$g(\pm 0.65) \simeq 0.10$$

CHECK:

$$g(0.65) \simeq P\left[z < \frac{0.392 - 0.65}{1/5}\right]$$
$$= P\left[z < -1.290\right]$$
$$= 1 - 0.9015$$
$$= 0.0985 \simeq 0.1$$

Pg 161

11) a) $H_0: \mu = \mu_0$

$H_1: \mu = \mu_1 > \mu_0$

α & β ARE GIVEN

$\alpha = P[\text{TYPE I ERROR GIVEN } \mu = \mu_0]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu = \mu_0]$

$= P[\bar{x} > d \text{ GIVEN } \mu = \mu_0]$ WHERE d = DECISION BOUNDRY

$= P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > \frac{d - \mu_0}{\sigma/\sqrt{n}}\right]$

$= P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > Z_\alpha\right]$ SINCE $\frac{d - \mu_0}{\sigma/\sqrt{n}} = Z_\alpha$

$= P[Z > Z_\alpha]$

$\beta = P[\text{TYPE II ERROR GIVEN } \mu = \mu_1 > \mu_0]$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = \mu_1 > \mu_0]$

$= P[\bar{x} < d \text{ GIVEN } \mu = \mu_1 > \mu_0]$

$= P\left[\frac{\bar{x} - \mu_1}{\sigma/\sqrt{n}} < \frac{d - \mu_1}{\sigma/\sqrt{n}}\right]$

$= P[Z < -Z_\beta]$ SINCE $\frac{\mu_1 - d}{\sigma/\sqrt{n}} = Z_\beta$

THUS:

$Z_\beta = -\frac{d - \mu_1}{\sigma/\sqrt{n}}$; $Z_\alpha = \frac{d - \mu_0}{\sigma/\sqrt{n}}$

SOLVING FOR d;

$d = -\frac{Z_\beta \sqrt{n}}{\sigma} + \mu_1 = \frac{Z_\alpha \sqrt{n}}{\sigma} + \mu_0$

ALGEBRAIC TRICKERY:

$\frac{\sigma}{\sqrt{n}}(Z_\alpha + Z_\beta) = \mu_1 - \mu_0$

SOLVING FOR $\sqrt{n}$

$\sqrt{n} = \sigma(Z_\alpha + Z_b)/(\mu_1 - \mu_0)$

SQUARING BOTH SIDES:

$$n = \frac{\sigma^2 (Z_\alpha + Z_b)^2}{(\mu_1 - \mu_0)^2}$$

11(b)? — $\frac{1}{2}$

Pg 170

1) $\mu_1 \rightarrow$ OLD AVERAGE
$\mu_2 \rightarrow$ NEW AVERAGE

a) $H_0 ; \mu_1 = \mu_2$
$H_1 ; \mu_2 > \mu_1$

b) $H_0 ; \mu_1 = \mu_2$
$H_1 ; \mu_1 > \mu_2$ (OR $\mu_2 < \mu_1$)

c) $H_0 ; \mu_1 = \mu_2$
$H_1 ; \mu_1 \neq \mu_2$

Pg 171

3) $n = 50$

$\bar{X} = 1.24$

$s \approx \sigma = 0.55$

$\mu = 1.40$

$\alpha = 0.05$

$H_0: \mu = 1.40$

$H_1: \mu < 1.24$

$$Z = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{1.24 - 1.40}{0.55/\sqrt{50}} = -2.06$$

$$-Z_\alpha = -Z_{0.05} = -1.645$$

$Z < Z_\alpha \Rightarrow$ REJECT H_0

YES

Pg 171

5) 20
19
22
17
18

$\bar{X} = 19.2$; $s^2 = 3.7$; $s = 1.9$; $n = 5$

$\alpha = 0.01$

$H_0: \mu = 22$

$H_1: \mu \neq 22$

$$t_{n-1;\alpha/2} = t_{4;0.005} = 4.604$$

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{19.2 - 22}{1.9/\sqrt{5}} = -3.26$$

IS $t < -t_{n-1;\alpha/2}$? (IS $-3.26 < -4.604$?) NO

IS $t > t_{n-1;\alpha/2}$? (IS $-3.26 > 4.604$) NO

SO ACCEPT H_0. (EVIDENCE IS NOT SUFFICIENT TO REJECT H_0)

ONE MUST ASSUME SAMPLING FROM A NORMAL POPULATION

Pg 171

7) $H_0 : \mu = 2000$

$H_1 : \mu < 2000$

$\alpha = 0.01$

$n = 6$

$\bar{x} = 1970$

$s = 6.8$

$-t_{n-1;\alpha/2} = t_{5;0.005} = ^{-}4.032$

$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{1970-2000}{6.8/\sqrt{6}} = ^{-}1.95$

$^{-}1.95 > ^{-}4.032$

MAY NOT REJECT NULL HYPOTHESIS

Pg 172

11) $\mu_R = 0.249$ $\quad\quad$ $\mu_S = 0.255$

$\sigma_R = 3 \times 10^{-3}$ $\quad\quad$ $\sigma_S = 2 \times 10^{-3}$

a) $\mu_T = \mu_S - \mu_R = 0.255 - 0.249$

$= 0.006$

$\sigma_T = \sqrt{\sigma_R^2 + \sigma_S^2}$

$= \left[(2)^2 + (3)^2\right]^{1/2} \times 10^{-3}$

$= 3.61 \times 10^{-3}$

b) $H_0: \mu_S - \mu_R = 0$

$H_1: \mu_S - \mu_R < 0$

FIND: $P\left[\bar{X}_S - \bar{X}_R = \bar{X} < 0\right]$

$P[\bar{X} < 0] = P\left[\frac{\bar{X} - \mu}{\sigma} < -\frac{\mu}{\sigma}\right]$

$= P\left[Z < -\frac{6 \times 10^{-3}}{3.61 \times 10^{-3}}\right]$

$= P[Z < -1.66]$

$= 0.485$

Pg 172

13) $\bar{X}_1 = 58$ $\quad \bar{X} = 66$

$n_1 = 10$ $\quad n_2 = 12$

$s_1 = 6$ $\quad s_2 = 4$

$H_0: \mu_2 - \mu_1 = 0$

$H_1: \mu_1 - \mu_2 < 0$

$\alpha = 0.01$

$$t = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{s\sqrt{1/n_1 + 1/n_2}}$$

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(9)(36) + (11)(16}{8}$$

$$= 25$$

$$\Rightarrow s = 5$$

THEN: $t = \dfrac{-8}{5\sqrt{1/10 + 1/12}} = -3.72$

$-t_{n_1+n_2-2;\alpha} = -t_{8;0.01} = -2.90$

$t_{n_1+n_2-2;\alpha} > t$

SO REJECT H_0

Pg 171

15) $n = 50$

$\bar{X}_1 = 120$ $\quad$ $\bar{X}_2 = 112$ $\quad$ $H_0: \mu_1 - \mu_2 = 0$

$S_1 = 12$ $\quad$ $S_2 = 9$ $\quad$ $H_1: \mu_1 - \mu_2 \neq 0$

$\alpha = 0.05$

$$S^2 = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}$$

$$= \frac{49(144 + 81)}{98}$$

$$= 112.5 \Rightarrow S = 10.7$$

$$t = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{S\sqrt{1/n_1 + 1/n_2}}$$

$$= \frac{8}{10.7\sqrt{1/25}}$$

$$= 3.75$$

$$t_{n_1+n_2-2;\alpha/2} = t_{98;\alpha/2}$$

$$\simeq Z_{\alpha/2}$$

$$= Z_{0.025}$$

$$= 1.960$$

$$t > -t_{n_1+n_2-2;\alpha/2} \quad (\text{ie } 1.96 > -3.75)$$

SO REJECT H_0

Pg 172

14) $n_1 = 10$ $n_2 = 8$

$\bar{x}_1 = 24$ $\bar{x}_2 = 30$

$s_1 = 16$ $s_2 = 25$

$\alpha = 0.01$

$H_0 : \mu_1 - \mu_2 = 0$

$H_1 : \mu_1 - \mu_2 \neq 0$

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 1}$$

$$= 19.9 \Rightarrow s = 4.46$$

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s\sqrt{1/n_1 + 1/n_2}}$$

$$= \frac{-6}{4.46\sqrt{1/10 + 1/8}}$$

$$= -2.83$$

WITH BURDEN OF PROOF ON A

$$t_{n_1 + n_2 - 2; \alpha/2} = t_{16; 0.005} = 2.921$$

ACCEPT H_0 (WITH CAUTION)

Pg 174

20) $n_a = 10$ $n_b = 8$ $\alpha = 0.01$

$\bar{X}_a = 24$ $\bar{X}_b = 30$

$S_a^2 = 16$ $S_b^2 = 25$

$S_a = 4$ $S_b = 5$

$H_0;\ \mu_1 - \mu_2 = \delta = 0$

$H_1;\ \mu_1 - \mu_2 \neq \delta = 0$

$$t' = \frac{(\bar{X}_1 - \bar{X}_2) - \delta}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

ν = DEGREES OF FREEDOM

$$= \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}}$$

$$= \frac{\left(\frac{16}{10} + \frac{25}{8}\right)^2}{\frac{(16/10)^2}{9} + \frac{(25/8)^2}{7}}$$

$$= \frac{22.3}{1.68} = 13.3$$

$$t = \frac{(24 - 30)}{\sqrt{\frac{16}{10} + \frac{25}{8}}} = \frac{-6}{2.18} = -2.76$$

$$t_{\nu;\alpha/2} \simeq -t_{13;5\times10^{-3}} = -3.012 < -2.76$$

$\therefore$ ACCEPT H_0

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{1/n_1 + 1/n_2}}$$

$$s = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$\nu = n_1 + n_2 - 2$$

$$s^2 = \frac{(9)(16) + (7)(25)}{16}$$

$$= \frac{144 + 175}{16} = 19.94 \Rightarrow s = 4.46$$

$$t = \frac{-6}{4.46\sqrt{0.1 + 0.125}}$$

$$= \frac{-6}{(4.46)(0.75)} = -2.83$$

$$-t_{n_1+n_2-2;\,.005} = -t_{16;\,.005} = -2.921 < -2.83$$

$\therefore$ ACCEPT H_0

Pg 179

3) $n = 5$

$\underline{X_i}$; 20, 19, 22, 17, 18

$\bar{x} = 19.2$; $S = 1.9$; $S^2 = 3.7$

$\alpha = 0.05$

FIRST FIND $\chi^2_{n-1;\alpha/2} = \chi^2_{4;0.025} = 11.14$

$$\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}} = \frac{(4)(3.7)}{11.14} = 1.325$$

FIND:

$$\chi^2_{n-1;1-\alpha/2} = \chi^2_{4;0.975} = 0.484$$

$$\frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}} = \frac{(4)(3.7)}{0.484} = 30.5$$

$\therefore \quad 1.325 < \sigma^2 < 30.5$

$1.15 < \sigma < 5.53$

Pg 179

5) $n = 25$

$\bar{x} = 175.36$

$s = 16.94$; $\alpha = 0.05$

a) $\chi^2_{24;0.025} = 39.4$

$$\frac{(n-1)s^2}{\chi^2_{24;0.025}} = \frac{(24)(16.9)^2}{39.4}$$

$$= 175$$

$\chi^2_{24:0.975} = 12.4$

$$\frac{(n-1)s^2}{\chi^2_{24:0.975}} = \frac{(24)(16.9)^2}{39.4}$$

$$= 555$$

$\Rightarrow 175 < \sigma^2 < 555$

$\therefore 13.2 < \sigma < 23.5$

b) $z_{\alpha/2} = z_{0.025}$
$= 1.96$

$$\frac{s}{1 + z_{\alpha/2}/\sqrt{2n}} = \frac{16.94}{1 + 1.96/\sqrt{50}} = 13.3$$

$$\frac{s}{1 - z_{\alpha/2}/\sqrt{2n}} = \frac{16.94}{1 - 1.96/\sqrt{50}} = 23.5$$

PRETTY CLOSE

Pg 183

1) $n = 9$

$\bar{x} = 84$

$s = 1.2$

$\alpha = 0.05$

$H_0 ; \sigma = 1.5$

$H_1 ; \sigma < 1.5$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$
$$= \frac{(8)(1.2)^2}{(1.5)^2}$$
$$= 5.12$$

$$\chi^2_{1-\alpha} = \chi^2_{0.95}$$
$$= 2.73$$

CANNOT REJECT H_0

Pg 197

1) $n = 200$

$X = 156$

$\alpha = 0.05$

$H_0 : p = 0.70$

$H_1 : p > 0.70$ $\Rightarrow p_0 = 0.70$

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}}$$
$$= \frac{156 - 200(0.70)}{\sqrt{200(0.70)(0.30)}}$$
$$= 2.47$$

$Z_\alpha = Z_{0.05} = 1.64$

is $Z > Z_\alpha$

$2.47 > Z_{0.05} = 1.64 \Rightarrow$ REJECT H_0

Pg 197

1) $n = 200$

$X = 156$

$\alpha = 0.05$

$H_0: p = 0.70$

$H_1: p > 0.70 \qquad \Rightarrow p_0 = 0.70$

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}}$$

$$= \frac{156 - 200(0.70)}{\sqrt{200(0.70)(0.30)}}$$

$$= 2.47$$

$Z_\alpha = Z_{0.05} = 1.64$

IS $Z > Z_\alpha$

$2.47 > Z_{0.05} = 1.64 \Rightarrow$ REJECT H_0

Pp. 197-8

4) $n = 400$

$X = 73$

$p_0 = 0.3$

$\alpha = 0.01$

$H_0: p = 0.3$

$H_1: p \neq 0.3$

$$z = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}} = \frac{73 - 400(0.3)}{\sqrt{400(0.3)(0.7)}} = {}^{-}5.13$$

$-z_{0.005} = {}^{-}2.576$

$z < -z_{\alpha/2} \Rightarrow$ REJECT H_0

Pg 198

5) $p_0 = 0.3$

$n = 500$

$x = 115$

$\alpha = 0.01$

$H_0: \; p = p_0 = 0.3$

$H_1: \; p \neq p_0$

$$z = \frac{x - np_0}{\sqrt{np_0(1-p_0)}}$$

$$= \frac{115 - (0.3)(500)}{\sqrt{500(0.3)(0.7)}}$$

$$= -3.13$$

$z_{\alpha/2} = z_{0.005} \approx 2.58$

$z < -z_{\alpha/2} \Rightarrow$ REJECT H_0

Pg 198

10) $n_1 = 80 \qquad X_1 = 25$

$n_2 = 50 \qquad X_2 = 21$

$\alpha = 0.05$

$H_0: p_1 = p_2 \qquad H_1: p_1 \neq p_2$

a) $\chi^2 = \sum_{i=1}^{2} \frac{(x_i + n_i \hat{p})}{n_i \hat{p}(1-\hat{p})}$

$\hat{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{46}{130} = 0.354$

$\Rightarrow \chi^2 = \frac{(25 - 28.3)^2}{80(0.354)(0.646)} + \frac{(21 - 17.7)^2}{50(0.354)(0.646)}$

$= 6 + 0.96 = 1.56$

$\chi^2_{1;0.05} = 3.841$

$\chi^2 < \chi^2_{1;0.05}$

$\Rightarrow$ CANNOT REJECT H_0

b) $z = \frac{x_1/n_1 - x_2/n_2}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}}$

$= -1.24$

$z_{0.025} = -1.96$

CANNOT REJECT H_0

Pg. 199

13)

SUCCESS	$\underline{13.4}$ 11	$\underline{10.95}$ 16	$\underline{17.6}$ 15	$\Rightarrow 42 = \sum f_{1j}$
FAILURE	$\underline{91.6}$ 94	$\underline{75.05}$ 70	$\underline{120.4}$ 123	$\Rightarrow 287 = \sum f_{2j}$
	105	86	138	329

$H_0: p_1 = p_2 = p_3$

$e_{11} = \dfrac{105 \times 42}{329} = 13.4$

$e_{12} = \dfrac{86 \times 42}{329} = 10.95$

$e_{13} = \dfrac{42 \times 138}{329} = 17.6$

$e_{21} = \dfrac{105 \times 42}{329} = 91.6 = n_1 - e_{11}$ etc.

$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{1}{e_{ij}} (f_{ij} - e_{ij})^2$$

$= 3.078$

$\chi^2_{2;0.05} = 5.991$

CANNOT REJECT H_0

Pg 203

1)

$\boxed{10.6}$ 9	$\boxed{11.2}$ 15	$\boxed{10.2}$ 8	$n_1 = 31 \Rightarrow f_{1j}$
$\boxed{25.1}$ 18	$\boxed{26.5}$ 31	$\boxed{24.4}$ 27	$n_2 = 76 \Rightarrow f_{2j}$
$\boxed{53.3}$ 62	$\boxed{56.3}$ 48	$\boxed{51.4}$ 51	$n_3 = 161 \Rightarrow f_{3j}$
$n_{.1} = 89$	$n_{.2} = 94$	$n_{.3} = 86$	$n = 269$

$$e_{11} = \frac{n_1 n_{.1}}{n} = 10.6$$

$$e_{12} = \frac{n_1 n_{.2}}{n} = 11.2$$

ETC,

$H_0: p_{i1} = p_{i2} = p_{i3} \ni i = 1, 2, 3$

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

$$= \frac{(9-10.6)^2}{10.6} + \frac{(11.2-15)^2}{15} + \frac{(8-10.2)^2}{8} + \text{ETC}$$

$$= 7.702$$

$$\chi^2_{(k-1)(r-1):\alpha} = \chi^2_{4;0.05} = 9.48$$

CANNOT REJECT H_0

Pg 204

3)

	A	B	C	D	
	(11) $\boxed{6}$ 7	(12) $\boxed{8}$ 5	(13) $\boxed{9\frac{1}{3}}$ 12	(14) $\boxed{6\frac{2}{3}}$ 6	30
	(21) $\boxed{16.6}$ 18	(22) $\boxed{22.1}$ 32	(23) $\boxed{25.8}$ 25	(24) $\boxed{18.4}$ 8	83
	(31) $\boxed{67.5}$ 65	(32) $\boxed{90}$ 83	(33) $\boxed{105}$ 103	(34) $\boxed{75}$ 86	337
	90	120	140	100	450

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{1}{e_{ij}} (e_{ij} - f_{ij})^2$$

$$= 14.865$$

$$\chi^2_{6;\,0.05} = 12.59$$

$\Rightarrow$ REJECT $H_0: p_{i1} = p_{i2} = p_{i3}$

Pg 217

2) $H_0: \mu = 100$

$H_1: \mu > 100$

+	−	+	−	+	+	+	−
+	0	+	−	+	+	−	+

10(+)'s ; 5(−)'s ; $n = 15$

TEST $H_0: p = 1/2$

$x = 10$

$n = 15$

$p_0 = 1/2$

$$Z = \frac{x - np_0}{\sqrt{np_0(1-p_0)}}$$

$$= \frac{10 - 7.5}{\sqrt{15/4}} = 1.29$$

$Z_\alpha = Z_{0.05} = 1.645$

$Z < Z_\alpha \Rightarrow$ CANNOT REJECT H_0

Pg 218

7) H_0: YARN A & YARN B ARE FROM SAME POP

H_1: THE POP. HAVE DIFFERENT MEANS

$\alpha = 0.05$

YARN	B	B	B	A	B		
OBS.	140.8	140.9	142.4	143.0	143.0		
RANK	1	2	3	4.5	4.5		
YARN	B	A	A	B	A	A	A
OBS	143.2	143.6	144.0	144.4	144.8	144.8	144.8
RANK	6	7	8	9	11	11	11
YARN	A	A	A	A	A	B	B
OBS.	145.2	145.5	145.6	145.6	146.0	146.6	146.8
RANK	13	14	15.5	15.5	17	18	19
YARN	A	B	B	B	B		
OBS	147.4	147.8	148.8	150.6	153.0		
RANK	20	21	22	23	24		

$R_1 = 1+2+3+4.5+6+9+18+19+21+22+23+24$

$= 152.5$

$n_1 = 12$; $n_2 = 12$

$\mu_U = \frac{n_1 n_2}{2} = 72$; $\sigma_U^2 = \frac{n_1 n_2 (2 n_1 n_2 - n_1 n_2)}{(n_1+n_2)^2 (n_1+n_2-1)} = 300$

$U = 144 + \frac{12(13)}{2} - 152.5 = 69.5$

$Z = \frac{U - \mu_U}{\sigma_\mu} = -0.1445$

$-Z_{\alpha/2} = -1.96$

$\Rightarrow$ CANNOT REJECT H_0

Pg 223-4

2) RUN TEST $\qquad$ H_0: RANDOM DATA

$n_1\,(O) = 20$

$n_2\,(N) = 80$

$\mu_u = \frac{2(80)(20)}{100} + 1 = 33$

$\sigma_\mu^2 = 10 \Rightarrow \sigma_\mu = 3.16$

$Z = \frac{23-33}{3.16} = -3.16$

($\mu = 23$ RUNS)

$Z = \frac{23-33}{3.16} = -3.16$

$-Z_{\alpha/2} = -1.96$

REJECT H_0

Pg 224

4) H_0: DATA IS A R.S.

H_1: DATA CONTAINS A TREND (LOWER TAIL)

MEDIAN = 32

n_1 = (BELOW MEDIAN) = 24

n_2 = (ABOVE MEDIAN) = 26

μ = # OF RUNS = 12

$\mu_\mu = 26$

$\sigma_\mu^2 = 5.86 \Rightarrow \sigma_\mu = 2.42$

$Z = \dfrac{\mu - \mu_\mu}{\sigma_\mu} = -5.78$

$-Z_\alpha = -1.645$

$\Rightarrow$ REJECT H_0

Pg 224

5) MEDIAN = 270.5

H_0: DATA IS RANDOM

H_1: THERE IS A CYCLE (UPPER TAIL TEST)

n_1 (ABOVE MEDIAN) = 12

n_2 (BELOW MEDIAN) = 12

μ = # OF RUNS = 13

$$\mu_u = \frac{2n_1 n_2}{n_1 + n_2} + 1 = 13$$

$$\sigma_\mu^2 = 0.957$$

$Z = 0$

ACCEPT H_0

Pg. 237

3)

x	1	2	3	4	5	6
Y	15	35	41	63	77	84

$n=6$

$\Sigma X_i = 21 \qquad \Sigma X_i^2 = 91 \qquad \Sigma X_i Y_i = 1349$

$\Sigma Y_i = 315 \qquad \Sigma Y_i^2 = 20085$

$\therefore \bar{X} = \frac{\Sigma X_i}{n} = 3.5 \quad ; \quad \bar{Y} = \frac{\Sigma Y_i}{n} = 52.5$

$\Sigma (x_i - \bar{X})^2 = 17.5$

$SST = \Sigma Y_i^2 - n\bar{Y}^2 = 3547.5$

$\Sigma (X_i - \bar{X})(Y_i - \bar{Y}) = \Sigma X_i Y_i - n\bar{X}\bar{Y} = 246.5$

$\therefore b = \frac{\Sigma (X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma (X_i - \bar{X})^2} = 14.086$

$a = \bar{Y} - b\bar{X} = 3.2$

$SSR = b^2 \Sigma (X_i - \bar{X})^2 = 3472$

$SSD = SST - SSR = 75.4$

$S_e^2 = \frac{SSD}{n-2} = 18.8$

$S_b^2 = \frac{S_e^2}{\Sigma (X_i - \bar{X})^2} = 1.08 \Rightarrow S_b = 1.04$

$b \pm t_{n-2;\alpha/2} S_b \Rightarrow t_{n-2;\alpha/2} = t_{4;0.25} = 2.776$

$\Rightarrow 14.086 \pm 2.881$

$\Rightarrow P(11.205 < \beta < 16.967) = 0.95$

Pg 237

3) $\sum X_i = 21 \quad \sum X_i^2 = 91 \quad \sum X_i Y_i = 1349$

$\sum Y_i = 315 \quad \sum Y_i^2 = 20085$

$b = \dfrac{\sum X_i Y_i - n\bar{X}\bar{Y}}{\sum X_i - n\bar{X}^2} = 14.086$

$a = \bar{Y} - b\bar{X} = 3.2$

$SSR = b^2 \sum (X_i - \bar{X})^2 = 3472$

$SST = \sum Y_i^2 - n\bar{Y}^2$

$= 20085 - 6\left(\frac{315}{6}\right)^2$

$= 3547.5$

$SSD = SST - SSR$

$= 75.4$

$S_e^2 = SSD/n-2 = 75.4$

$S_a^2 = \dfrac{S_e^2 \sum X_i}{n \sum (X_i - \bar{X}^2)}$

$S_b^2 = S_e^2 / \sum (X_i - \bar{X})^2 = 1.08 \Rightarrow S_b = 1.04$

$t_{n-2;\alpha/2} = t_{4;0.025} = 2.776$

OR 14.086 ± 2.881

BOB MARKS

8/8

Pg. 172; 174

14) $n_1 = 10 \quad \bar{X}_1 = 24 \quad S_1^2 = 16 \Rightarrow S_1 = 4$

$n_2 = 8 \quad \bar{X}_2 = 30 \quad S_2^2 = 25 \Rightarrow S_2 = 4$

$H_0 : \mu_1 = \mu_2$ (ie $\mu_1 - \mu_2 = 0$)

$H_1 : \mu \neq \mu_2$ (ie $\mu_1 - \mu_2 \neq 0$

$\alpha = 0.01$ (ASSUME $\sigma_1^2 = \sigma_2^2$)

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{1/n_1 + 1/n_2}}$$

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(9)(16) + (7)(25)}{16} = \frac{144 + 175}{16} = \frac{319}{16} = 19.9$$

$\therefore s = 4.46$

$$t = \frac{24 - 30}{4.46\sqrt{1/10 + 1/8}} = \frac{-6}{4.46\sqrt{0.1 + 0.125}} = \frac{-6}{4.46\sqrt{0.225}}$$

$$= \frac{-6}{4.46 \times 0.475} = -2.83$$

$$-t_{n_1+n_2-2;\alpha/2} = -t_{16;0.005} = -2.921$$

$t > -t_{n_1+n_2-2;\alpha/2} \Rightarrow$ WE CANNOT REJECT H_0

20) IF $\sigma_1^2 \neq \sigma_2^2$ (MAYBE)

$$t = (\bar{X}_1 - \bar{X}_2)\Big/\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

$$= -6\Big/\sqrt{\frac{16}{10} + \frac{25}{8}}$$

$$= -6\Big/\sqrt{1.6 + 3.13}$$

$$= -6\Big/\sqrt{4.73} = -6/2.18 = -2.76$$

$$z = \text{D.O.F} = \left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2 \Big/ \left[\frac{(S_1^2/n_1)^2}{n_1 - 1} + \frac{(S_2^2/n_2)^2}{n_2 - 1}\right]$$

$$\frac{\left(\frac{16}{10} + \frac{25}{8}\right)^2}{\frac{(16/10)^2}{9} + \frac{(25/8)^2}{7}}$$

$$= \frac{(4.73)^2}{\frac{(1.6)^2}{9} + \frac{(3.13)^2}{7}} = \frac{22.4}{\frac{2.53}{9} + \frac{9.80}{7}} = \frac{22.4}{0.285 + 1.4}$$

$$= \frac{22}{1.69} = 13.0$$

$$-t_{r;\alpha/2} = -t_{13;0.005} = -3.012$$

$\Rightarrow$ WE CANNOT REJECT H_0

ANSWERS SIMILAR TO PROB. 14.
IN NOT ASSUMING EQUAL POPULATION
VARIANCES, WE LOST 3 D.O.F.

GIVEN $\quad -z_{\alpha/2} \le \dfrac{x - np}{\sqrt{np(1-p)}} \le z_{\alpha/2}$

EQUIVALENTLY:

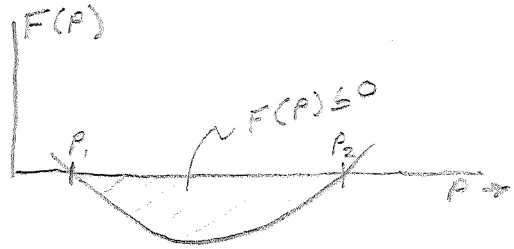
$$\left|\frac{x-np}{\sqrt{np(1-p)}}\right| \le z_{\alpha/2}$$

$$\frac{(x-np)^2}{np(1-p)} \le z_{\alpha/2}^2$$

$$(x-np)^2 - z_{\alpha/2}^2\, np(1-p) \le 0$$

$$x^2 - 2xnp + n^2p^2 - z_{\alpha/2}^2\, np + z_{\alpha/2}\, np^2 \le 0$$

$$F(p) = (n^2 + z_{\alpha/2}^2 n)\,p^2 - (2xn + z_{\alpha/2}^2 n)\,p + x^2 \le 0$$



USE QUADRATIC FORMULA:

$$p_c = \frac{n(2x + z_{\alpha/2}^2) \pm \sqrt{n^2(2x + z_{\alpha/2}^2)^2 - n4x^2(n + z_{\alpha/2}^2)}}{2n(n + z_{\alpha/2}^2)}$$

$$= \frac{n(2x + z_{\alpha/2}^2) \pm \sqrt{4\cancel{x^2}n^2 + 4n^2x z_{\alpha/2}^2 + n^2 z_{\alpha/2}^4 - \cancel{n4x^2n} - n4x^2 z_{\alpha/2}^2}}{2n(n + z_{\alpha/2}^2)}$$

$$= \frac{n(2x + z_{\alpha/2}^2) \pm \sqrt{4n^2x z_{\alpha/2}^2 + n^2 z_{\alpha/2}^4 - n4x^2 z_{\alpha/2}^2}}{2n(n + z_{\alpha/2}^2)}$$

$$= \frac{x + \frac{1}{2} z_{\alpha/2}^2 \pm \sqrt{x z_{\alpha/2}^2 + \frac{1}{4} z_{\alpha/2}^4 - \frac{x^2 z_{\alpha/2}^2}{n}}}{(n + z_{\alpha/2}^2)}$$

$$= \frac{x + \frac{1}{2} z_{\alpha/2}^2 \pm z_{\alpha/2}\sqrt{x - \frac{x^2}{n} + \frac{1}{4} z_{\alpha/2}^2}}{n + z_{\alpha/2}^2}$$

$$= \frac{x + \frac{1}{2} z_{\alpha/2}^2 \pm z_{\alpha/2}\sqrt{\frac{x(n-x)}{n} + \frac{1}{4} z_{\alpha/2}^2}}{n + z_{\alpha/2}^2}$$

Pg 198

7) $\chi^2 = \dfrac{(x_1 - n_1\hat{p})^2}{n_1\hat{p}(1-\hat{p})} + \dfrac{(x_2 - n_2\hat{p}_2)^2}{n_2\hat{p}(1-\hat{p})}$

$= \dfrac{n_2(x_1 - n_1\hat{p})^2 + n_1(x_2 - n_2\hat{p})^2}{n_1 n_2 \hat{p}(1-\hat{p})}$

$= \dfrac{(n_1+n_2)[(x_1 - n_1\hat{p})^2 n_2 + n_1(x_2 - n_2\hat{p})^2]}{n_1 n_2 (n_1+n_2)\hat{p}(1-\hat{p})}$

NUM $= (n_1+n_2)[(x_1 - n_1 p)^2 n_2 + (x_2 - n_2\hat{p})^2 n_1]$

$= (n_1+n_2)[(x_1^2 - 2n_1x_1\hat{p} + n_1^2\hat{p}^2)n_2 + (x_2^2 - 2n_2x_2\hat{p} + n_2^2\hat{p}^2)n_1]$

$= (n_1+n_2)[n_2x_1^2 - 2n_1n_2x_1\hat{p} + n_1^2n_2\hat{p}^2$

$+ n_1x_2^2 - 2n_1n_2x_2\hat{p} + n_1n_2^2\hat{p}^2]$

$= n_1n_2x_1^2 - 2n_1^2n_2x_1\hat{p} + \underline{n_1^3n_2}\hat{p}^2$

$+ \underline{n_2^2x_1^2} - 2n_1n_2^2x_1\hat{p} + n_1^2n_2^2\hat{p}^2$

$+ \underline{n_1^2x_2^2} - 2n_1^2n_2x_2\hat{p} + \underline{n_1^2n_2^2}\hat{p}^2$

$+ n_1n_2x_2^2 - 2n_1n_2^2x_2\hat{p} + \underline{n_1n_2^3}\hat{p}^2$

$= n_2^2x_1^2 + n_1^2x_2^2 + (n_1^3n_2 + 2n_1^2n_2^2 + n_1n_2^3)\hat{p}^2$

$+ n_1n_2x_1^2 - 2n_1^2n_2x_1\hat{p} - 2n_1n_2^2x_1\hat{p} - 2n_1^2n_2x_2\hat{p}$

$+ n_1n_2x_2^2 - 2n_1n_2^2x_2\hat{p}$

$= n_2^2x_1^2 + n_1^2x_2^2 + \hat{p}^2 n_1n_2(n_1^2 + 2n_1n_2 + n_2^2)$

$+ \underline{n_1n_2x_1^2} - \underline{2n_1^2n_2x_1\hat{p}} - 2n_1n_2^2x_1\hat{p} - 2n_1^2n_2x_2\hat{p}$

$+ n_1\underline{n_2}x_2^2 - 2n_1n_2^2x_2\hat{p}$

$= n_2^2x_1^2 + n_1x_2^2 + (x_1 + x_2^2)n_1n_2$

$- 2n_1n_2\hat{p}(n_1x_1 + n_2x_1 + n_1x_2 + n_2x_2)$

$+ n_1n_2(x_1^2 + x_2^2)$

$= n_2^2x_1^2 + n_1^2x_2^2 + 2n_1n_2(x_1^2 + x_2^2)$

$- 2n_1n_2\hat{p}[x_1(n_1+n_2) + x_2(n_1+n_2)]$

$$= n_2^2 x_1^2 + n_1^2 x_2^2 + 2n_1 n_2 (x_1^2 + x_2^2) - 2n_1 n_2 (x_1 + x_2)^2$$

$$= n_2^2 x_1^2 + n_1^2 x_2^2 + 2n_1 n_2 (x_1^2 + x_2^2) - 2n_1 n_2 (x_1^2 + 2x_1 x_2 + x_2^2)$$

$$= n_2^2 x_1^2 - 2n_1 n_2 x_1 x_2 + n_1^2 x_2^2 + 2n_1 n_1 (x_1^2 + x_2^2) - 2n_1 n_2 (x_1^2 + x_2^2)$$

$$= n_2^2 x_1^2 - 2n_1 n_2 x_1 x_2 + n_1^2 x_2^2$$

$$= (x_1 n_2 - x_2 n_1)^2$$

$$\Rightarrow \chi^2 = \frac{(x_1 n_2 - x_2 n_1)^2}{n_1 n_2 (n_1 + n_2) \hat{p}(1 - \hat{p})}$$

$$= \frac{(x_1 n_2 - x_2 n_1)^2 \frac{1}{n_1^2 n_2^2}}{\hat{p}(1 - \hat{p}) \left(\frac{n_1 + n_2}{n_1 n_2}\right)}$$

$$= \frac{\left(\frac{x_1}{n_1} - \frac{x_2}{n_2}\right)^2}{\hat{p}(1 - \hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = z^2$$

Pg 205

8) H_0: DATA FROM POISSON POP. WITH $\lambda = 3.4$

$\alpha = 0.1$

H_1: IT ISN'T

γ RAYS A SEC	f_i FREQ	POISSON PROB	e_i EXPECTED FREQ
0	3	0.033	8.25
1	21	0.114	28.5
2	51	0.193	48.25
3	60	0.218	54.5
4	38	0.186	46.5
5	31	0.127	31.75
6	26	0.071	17.75
≥ 7	20	0.058	14.5

$$\chi^2 = \sum_{i=1}^{8} \frac{(f_i - e_i)^2}{e_i}$$

$$= \frac{(3-8.15)^2}{8.15} + \frac{(21-28.5)^2}{28.5} + \frac{(51-48.25)^2}{48.25} + \frac{(60-54.5)^2}{54.5} + \frac{(38-46.5)^2}{46.5} + \frac{(31-31.75)^2}{46.5} + \frac{(31-31.75)^2}{31.75} + \frac{(26-17.75)^2}{17.75} + \frac{(20-14.6)^2}{14.6} = 13.3$$

$$r = 7 \Rightarrow \chi^2_{r;\alpha} = \chi^2_{7;0.1} = 18.5$$

$\Rightarrow$ WE CANNOT REJECT H_0

Pg 205

9)

0.10 - 0.29	3
0.30 - 0.49	13
0.50 - 0.69	17
0.70 - 0.89	32
0.90 1.09	23
1.10 - 1.29	7
1.30 - 1.49	5

$\bar{x} = 0.795$

$s = 0.28$

a) FIND $P[0.095 < y < 0.295]$

$= P\left[\frac{0.095-0.795}{0.28} < z < \frac{0.295-0.795}{0.28}\right]$

$= P[-2.5 < z < -1.79]$

$= 0.037 - 0.006 = 0.031$

$P[0.295 < y < 0.495] = P\left[\frac{0.295-0.795}{0.28} < z < \frac{0.495-0.795}{0.28}\right]$

$= P[-1.79 < z < -1.07]$

$=$

$= 0.1053$

$P[0.495 < y < 0.695] = P\left[\frac{0.495-0.795}{0.28} < z < \frac{0.695-0.795}{0.28}\right]$

$= P[-1.07 < z < -0.357]$

$= 0.2177$

$P[0.695 < y < 0.895] = P\left[\frac{0.695-0.795}{0.28} < z < \frac{0.895-0.795}{0.28}\right]$

$= P[-0.357 < z < 0.357]$

$= 0.28$

$$P[0.895 < y < 1.095] = P\left[\frac{0.895-0.795}{0.28} < Z < \frac{1.095-0.795}{0.28}\right]$$

$$= P[0.357 < Z < 1.07]$$

$$= \qquad = 0.218$$

$$P[1.095 < y < 1.295] = P\left[\frac{1.095-0.795}{0.28} < Z < \frac{1.295-0.795}{0.28}\right]$$

$$= P[1.07 < Z < 1.79]$$

$$= 0.105$$

$$P[1.295 < y < 1.495] = P\left[\frac{1.295-0.795}{0.28} < Z < \frac{1.495-0.795}{0.28}\right]$$

$$= P[1.79 < Z < 2.5]$$

$$= 0.0308$$

f	NORMAL POP	EXPECTED N( ) FREQ
3 } 16	0.0308	3.08 } combine
13	0.1053	10.5
17	0.2177	21.8
32	0.2800	28
23	0.2177	21.8
7 } 12	0.1053	10.5 } 13.6
5	0.0301	3.1

(EMPLOYING MILLER FREUND)

$$\chi^2 = \sum_{i=1}^{5} \frac{(f_i - e_i)^2}{e_i}$$

$$= \frac{(16-13.6)^2}{13.6} + \frac{(17-21.8)}{21.8} + \frac{(32-28)^2}{28} + \frac{(23-21.8)^2}{21.8} + \frac{(12-13.6)^2}{13.6}$$

$$= 2.30$$

$r = 5 - 3 = 2$

$\chi^2_{r;\alpha} = \chi^2_{2;0.05} = 5.991$

$\Rightarrow$ WE CANNOT REJECT H_0

Pg 205

10)

<10	35	H_0: DATA IS FROM AN EXPODENTIAL POPULATION WITH $\mu = 25$
10-20	20	H_1: IT ISN'T
20-30	18	$\alpha = 0.01$
30-40	8	
>40	19	
	100	

$\frac{1}{\mu} e^{-t/\mu}$

$$P(t_1 < t < t_2) = \int_{t_1}^{t_2} \frac{1}{\mu} e^{-t/\mu}\, dt$$

$$P(t < 10) = \int_0^{10} \frac{1}{\mu} e^{-t/\mu}\, dt = -e^{-\frac{10}{25}} + 1 = 0.330$$

$$P(10 < t < 20) = \int_{10}^{20} \frac{1}{\mu} e^{-t/\mu}\, dt = -e^{-\frac{20}{25}} + e^{-\frac{10}{25}} = 0.2210$$

$$P(20 < t < 30) = -e^{-\frac{30}{25}} + e^{-\frac{20}{25}} = 0.148$$

$$P(30 < t < 40) = -e^{-\frac{40}{25}} + e^{-\frac{30}{25}} = 0.0993$$

$$P(t > 40) = e^{-\frac{40}{25}} = 0.202$$

EXPECTED P	EXPECTED f
0.330	33.0
0.221	22.1
0.148	14.8
0.0993	9.93
0.202	20.2

$$\chi^2 = \frac{(35-33)^2}{33} + \frac{(20-22.1)^2}{22.1} + \frac{(18-14.8)^2}{14.8} + \frac{(8-9.93)}{9.93} + \frac{(19-20.2)^2}{20.2} = 1.45$$

$$\chi^2_{2,0.01} = 9.210$$

$\Rightarrow$ WE CAN DEFINATELY NOT REJECT H_0

Pg 219

10)

ATM.	3	3	3	5	2	3	3	5		
TEST	0.16	0.17	0.19	0.26	0.28	0.29	0.29	0.29		
RANK	1	2	3	4	5	7	7	7		
5	5	3	2	2	2	2	5	2	5	
0.30	0.34	0.34	0.35	0.37	0.37	0.41	0.42	0.44	0.47	
9	10.5	10.5	12	13.5	13.5	15	16	17	18	
4	1	2	1	1	1	4	1	4		
0.57	0.58	0.58	0.59	0.61	0.61	0.68	0.69	0.69		
19	20.5	20.5	22	23.5	23.5	25	26.5	26.5		
4	4	4	1	4	4					
0.72	0.75	0.77	0.79	0.81	0.85					
28	29	30	31	32	33					

$R_1 = 20.5 + 22 + 23.5 + 23.5 + 26.5 + 31 = 147 \Rightarrow R_1^2 = 2.16 \times 10^4$

$R_2 = 5 + 12 + 13.5 + 13.5 + 15 + 17 + 20.5 = 96.5 \Rightarrow R_2^2 = 9.31 \times 10^3$

$R_3 = 1 + 2 + 3 + 7 + 7 + 10.5 = 30.5 \Rightarrow R_3^2 = 930$

$R_4 = 19 + 25 + 26.5 + 28 + 29 + 30 + 32 + 33 = 222 \Rightarrow R_4^2 = 4.95 \times 10^4$

$R_5 = 4 + 7 + 9 + 10.5 + 16 + 18 = 64.5 \Longrightarrow R_5^2 = 4160$

$$\sum_{i=1}^{5} \frac{R_i^2}{n_i} = \frac{21600}{6} + \frac{9310}{7} + \frac{930}{5} + \frac{4950}{8} + \frac{4160}{6}$$

$$= 3600 + 1300 + 155 + 6190 + 694 = 11969$$

$$H = \frac{12}{n(n+1)} \sum R_i^2/n - 3(n+1) \quad ; \; n = 33$$

$$= \frac{12}{33(34)}(11969) - 3 \times 34 = 128 - 102 = 26$$

$$\chi^2_{4;0.05} = 9.488$$

REJECT H_0: THE FIVE SAMPLES WERE TAKEN FROM IDENTICAL POPULATIONS

Pg 198

12) $$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{2} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

$$= \sum_{j=1}^{k} \left[\frac{(f_{1j} - e_{1j})^2}{e_{1j}} + \frac{(f_{2j} - e_{2j})^2}{e_{2j}} \right]$$

$$f_{1j} = x_j \qquad f_{2j} = n_j - x_j$$
$$e_{1j} = n_j \hat{p} \qquad e_{2j} = n_j (1 - \hat{p})$$

$$\Rightarrow \chi^2 = \sum_{j=1}^{k} \left[\frac{(x_j - n_j \hat{p})^2}{n_j \hat{p}} + \frac{(n_j - x_j - n_j(1-\hat{p}))^2}{n_j(1-\hat{p})} \right]$$

$$= \sum_{j=1}^{k} \left[\frac{(x_j - n_j \hat{p})^2}{n_j \hat{p}} + \frac{(x_j - n_j \hat{p})^2}{n_j (1-\hat{p})} \right]$$

$$= \sum \frac{(x_j - n_j \hat{p})^2 \left[(1-\hat{p}) + \hat{p} \right]}{n_j \hat{p} (1-\hat{p})}$$

$$= \sum_{j=1}^{k} \frac{(x_j - n\hat{p})^2}{n_j \hat{p}(1-\hat{p})}$$

Pg 239

13) a) $a = \dfrac{\sum X_i^2 \sum Y_i - \sum X_i \sum X_i Y_i}{n \sum X_i^2 - (\sum X_i)^2}$; $\sum_{i=1}^{n}$

FOR DEPENDENT Y_i :

$\sum X_i^2 = C_1$

$\sum X_i = C_2 \ni C_1, C_2 \in$ CONSTANT

IT IS THUS POSSIBLE TO FACTOR Y_i :

$$a = \sum_{i=1}^{n} \left[\frac{\sum X_i^2 - (\sum X_i) X_i}{n \sum X_i^2 - (\sum X_i)^2} \right] Y_i$$

$$= \sum_{i=1}^{n} c_i Y_i \ni c_i \text{ IS A CONSTANT FOR EACH } X_i, Y_i \text{ PAIR}$$

b) $E[a] = E\left[\sum_{i=1}^{n} c_i Y_i\right]$

$E\left[\sum_{i=1}^{n} \xi_i\right] = \sum_{i=1}^{n} E[\xi_i]$; $E[cX] = cE[X]$

$$\Rightarrow E[a] = \sum_{i=1}^{n} c_i \, E[Y_i]$$

$$= \sum_{i=1}^{n} c_i (\alpha + \beta X_i)$$

$$= \alpha \sum_{i=1}^{n} c_i + \beta \sum_{i=1}^{n} c_i X_i$$

AGAIN: $\sum c_i = \sum \left[\dfrac{\sum X_i^2 - \sum (X_i) X_i}{n \sum X_i^2 - \sum X_i^2} \right]$

$$= \sum \frac{\sum X_i^2}{n \sum X_i^2 - (\sum X_i)^2} - \sum \frac{\sum (X_i) X_i}{n \sum X_i^2 - (\sum X_i)^2}$$

$$= \frac{n \sum X_i^2}{n \sum X_i^2 - (\sum X_i)^2} - \frac{(\sum X_i)^2}{n \sum X_i^2 - (\sum X_i)^2}$$

$$= \frac{n \sum X_i^2 - (\sum X_i)^2}{n \sum X_i^2 - (\sum X_i)^2} = 1$$

$$\sum c_i X_i = \sum \left[\frac{\sum X_i^2 - (\sum X_i) X_i}{n \sum X_i^2 - (\sum X_i)^2} \right] X_i$$

$$= \sum \frac{X_i \sum X_i^2 - (\sum X_i) X_i^2}{n \sum X_i^2 - (\sum X_i)^2}$$

$$= \frac{\sum X_i \sum X_i^2 - \sum X_i \sum X_i^2}{n \sum X_i^2 - (\sum X_i)^2} = 0$$

$$\Rightarrow E[a] = \alpha$$

Pg 239

14) a) $b = \dfrac{n \sum x_i Y_i - \sum x_i \sum Y_i}{n \sum x_i^2 - (\sum x_i)^2} \quad ; \quad \sum_{i=1}^{n}$

$= \sum \left[\dfrac{n x_i - \sum x_i}{n \sum x_i^2 - (\sum x_i)^2} \right] Y_i$

$= \sum C_i' Y_i \ni C_i'$ IS CONSTANT FOR EACH X_i, Y_i PAIR

b) $E[b] = E[\sum C_i' Y_i]$

$= \sum C_i' E[Y_i]$

$= \sum C_i' (\alpha + \beta x_i)$

$= \sum C_i' \alpha + \sum C_i' \beta X_i$

$= \alpha \sum C_i' + \beta \sum C_i' X_i$

$\sum C_i' = \sum \dfrac{n x_i - \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$

$= \dfrac{n \sum x_i - n \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$

$= 0$

$\sum C_i' x_i = \sum \dfrac{n x_i - \sum x_i}{n \sum x_i^2 - (\sum x_i)^2} x_i$

$= \dfrac{n \sum x_i^2 - (\sum x_i)^2}{n \sum x_i^2 - (\sum x_i)^2}$

$= 1$

$\Rightarrow E[b] = \beta$

$$\Rightarrow \quad b = -\frac{93.729}{2.5\times10^{6}} = -3.75\times10^{-5} \qquad \{E[b] = \beta\}$$

$$\Rightarrow b = -3.75\times10^{-5}$$

$$a' = \overline{Y'} + \beta\overline{X}$$

$$= 3.40$$

$$\Rightarrow a = e^{\alpha'} = 30 \qquad \{E[a] = \alpha\}$$

ESTIMATED FIT:

$$Y = 30\,e^{-3.75\times10^{-5}X}$$

Pg. 249

1) $Y = \alpha e^{-\beta x} \Rightarrow \ln Y = \ln \alpha - \beta X$

LET $Y' = \ln Y$; $\alpha' = \ln \alpha$

Y	$Y' = \ln Y$	X
29.9	3.398	0
29.4	3.381	500
29.0	3.367	1000
28.4	3.346	1500
27.7	3.321	2000

$n = 5$

$\sum_{i=1}^{5} X_i = 5000$; $\sum_{i=1}^{5} X_i^2 = 7500000. = 7.5 \times 10^6$

$\sum_{i=1}^{5} Y_i' = 16.814$; $\sum_{i=1}^{5} Y_i'^2 = 56.5455$

$\sum_{i=1}^{5} X_i Y_i' = 16720 = 1.672 \times 10^4$

$\sum_{i=1}^{5} (X_i - \overline{X})^2 = \Sigma X_i^2 - n \overline{X}^2$

$\overline{X} = \frac{1}{n} \Sigma X_i = 10^3$

$\Rightarrow \sum_{i=1}^{5} (X_i - \overline{X})^2 = 7.5 \times 10^6 - 5 \times 10^6$

$= 2.5 \times 10^6$

$SST = \sum_{i=1}^{5} (Y_i' - \overline{Y'}) = \Sigma Y_i'^2 - n \overline{Y'}^2$

$= 0.00356$

$= 3.56 \times 10^{-3}$

$\Sigma (X_i - \overline{X})(Y_i' - \overline{Y}) = \Sigma X_i Y_i' - n \overline{X}\, \overline{Y'}$

$= -93.729$

Pg. 250

2) a) $Y = \alpha B^{x} \Rightarrow \ln Y = \ln \alpha + x \ln B$

SO X IS PLOTTED LINEARY, Y ON ln (SEE GRAPH)

Y	$Y^* = \log_{10} Y$	X
96	1.985	1
75	1.875	5
63	1.799	10
30	1.477	25
9	0.954	50
2	0.301	100

$\Sigma X_i = 191$; $\Sigma X_i^2 = 13250 = 1.325 \times 10^4$

$\Sigma Y_i^* = 8.388$; $\Sigma Y_i^{*2} = 13.86$

$\Sigma X_i Y_i' = 1.441 \times 10^2$

$\Sigma (X_i - \overline{X})^2 = \Sigma X_i^2 - n\overline{X}^2 = 7.171 \times 10^3$

$\Sigma (X_i - \overline{X})(Y_i' - \overline{Y_i'}) = \Sigma X_i Y_i' - n\overline{X}\overline{Y} = -122.92$

$\therefore \log b = -1.714 \times 10^{-2}$

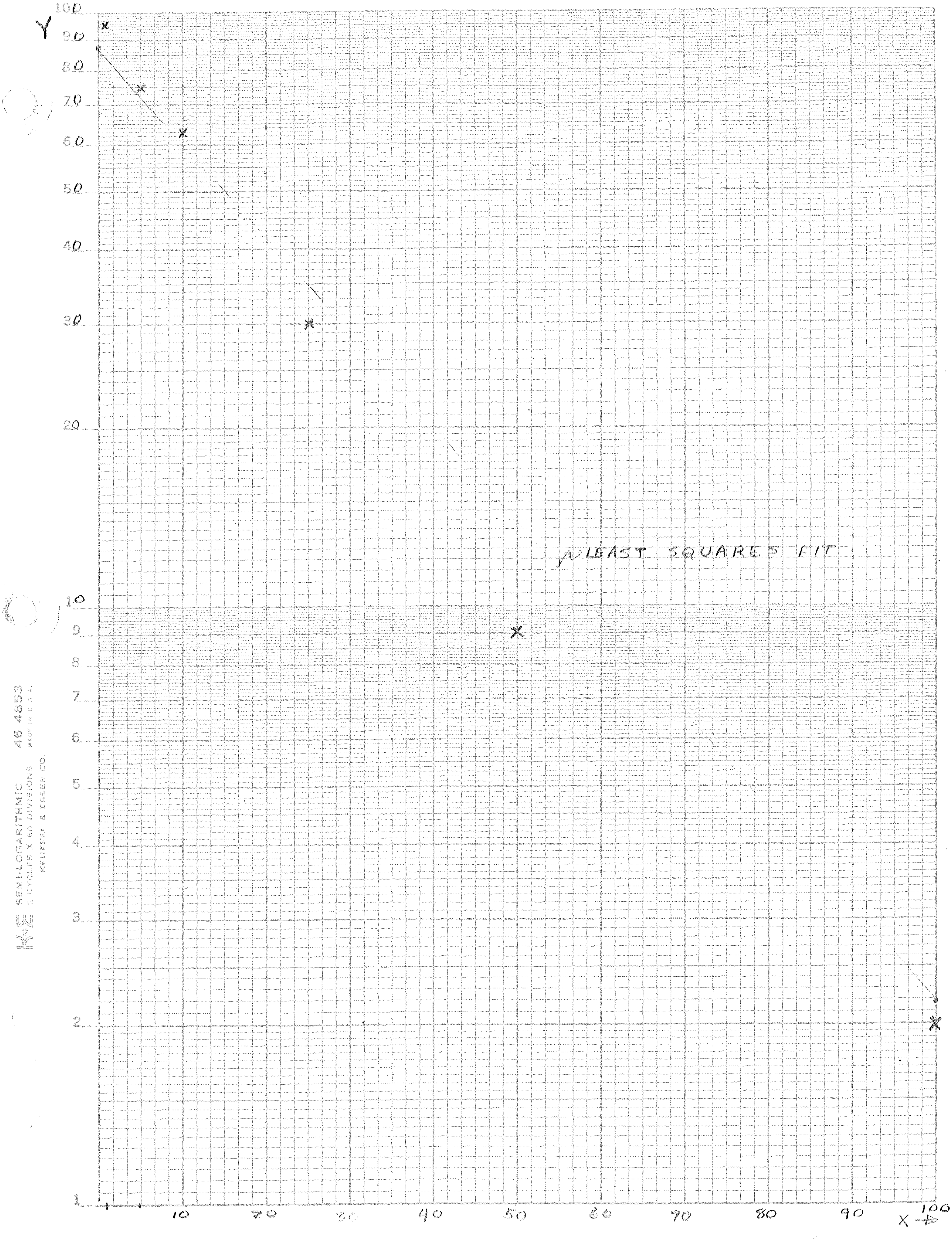
$\log a = \overline{Y} - \log B \, \overline{X}$

$= 1.944$

$\Rightarrow a = 87.84$; $b = 0.960$

$\therefore Y = 87.84 \, (0.960)^{X}$

b) $87.84 \, (0.960)^8 = 63.40$ GRAMS


Y
100
90
80
70
60
50
40
30
20
10
9
8
7
6
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4
3
2
1
LEAST SQUARES FIT
10
20
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70
80
90
100
X
K+E SEMI-LOGARITHMIC 46 4853
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Pp. 250-1

5) $PV^{\gamma} = C$

$\log P + \gamma \log V = \log C$

$\Rightarrow \log P = \log C - \gamma \log V$

Y	$\log Y = Y_L$	X	$\log X = X_L$
16.8	1.225	50	1.699
39.7	1.599	30	1.477
78.6	1.895	20	1.301
115.5	2.063	15	1.176
195.0	2.290	10	1.000
546.1	2.737	5	0.699

$\Sigma Y_{Li} = 11.809$ $\quad \Sigma Y_{Li}^2 = 11.809$

$\Sigma X_{Li} = 7.352$ $\quad \Sigma X_{Li}^2 = 9.632$

$\Sigma Y_{Li} X_{Li} = 13.538$ $\quad n = 6$

$\Rightarrow \overline{X}_L = 1.2253$ $\quad \overline{Y}_L = 1.9682$

$\Sigma (X_{Li} - \overline{X})^2 = \Sigma X_{Li}^2 - n\overline{X}_L^2 = 0.6234$

$SST = \Sigma (Y_{Li} - \overline{Y}_L)^2 = \Sigma Y_{Li}^2 - n\overline{Y}_L^2 = 1.3970$

$\Sigma (X_{Li} - \overline{X}_L)(Y_{Li} - \overline{X}_L) = \Sigma X_{Li} Y_{Li} - n\overline{X}\,\overline{Y} = -0.9320$

$\Rightarrow -\gamma = -1.495$

$\log C = \overline{Y}_L - \gamma \overline{X} = 3.80$

$\Rightarrow C = 6.310 \times 10^3$

PT ESTIMATE FOR γ IS 1.495

b) LET $b = -\gamma$; $\alpha = 0.05$

$b \pm t_{n-2;\frac{\alpha}{2}} S_b$

$SSR = b^2 \Sigma(x_i - \bar{X})^2 = 1.393$

$SSD = SST - SSR = 1.397 - 1.393 = 0.004$

$S_e^2 = \frac{SSD}{n-2} = 10^{-3}$

$S_b^2 = S_e^2 / \Sigma(x_i - \bar{X}_i)^2 = 1.60 \times 10^{-3}$

$\Rightarrow S_b = 0.04$

$t_{4;0.025} = 2.776$

$\Rightarrow t_{n-2;\alpha 2} S_b = 0.111$

$\therefore P[-1.606 \leq b \leq -1.384] = 0.95$

$P[1.384 \leq \gamma \leq 1.606] = 0.95$

Pg 251

8)

Y	X	Y	X
8.4	−12	11.2	1
7.6	−11	12.5	2
7.8	−10	12.2	3
8.3	−9	12.0	4
8.2	−8	12.1	5
8.8	−7	11.9	6
9.0	−6	12.0	7
8.8	−5	12.5	8
8.8	−4	12.9	9
9.1	−3	12.9	10
9.8	−2	12.2	11
10.7	−1	12.3	12
11.5	0		

$n = 25$

$\Sigma X_j = 0$

$\Sigma X_j^2 = 1300$

$\Sigma X_j^3 = 0$

$\Sigma X_j^4 = 1.2142 \times 10^3$

$\Sigma Y_j = 263.5$

$\Sigma Y_j X_j = 308.5$

$\Sigma Y_j X_j^2 = 1.3418 \times 10^4$

$$\begin{bmatrix} n & \Sigma x_j & \Sigma x_j^2 \\ \Sigma x_j & \Sigma x_j^2 & \Sigma x_j^3 \\ \Sigma x_j^2 & \Sigma x_j^3 & \Sigma x_j^4 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Sigma Y_j \\ \Sigma Y_j x_j \\ \Sigma Y_j x_j^2 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 0 & 1300 \\ 0 & 1300 & 0 \\ 1300 & 0 & 121420 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 263.5 \\ 308.5 \\ 13418 \end{bmatrix}$$

$$\Delta = 25 \times 1300 \times 121420 - (1300)^3 = 1.750 \times 10^9$$

$$\Delta_{b_0} = (263.5)(1300)(121420) - (1300)^2(13414)$$
$$= 1.890 \times 10^{10} \Rightarrow b_0 = \frac{1.89 \times 10^{10}}{1.75 \times 10^9} = 10.8$$

$$\Delta_{b_1} = 25(308.5)(121420) - (1300)^2(308.5)$$
$$= 4.15 \times 10^8 \Rightarrow b_1 = \frac{4.15 \times 10^8}{1.75 \times 10^9} = 0.237$$

$$\Delta_{b_2} = 25(1300)(13418) - (263.5)(1300)^2$$
$$= -9.22 \times 10^6 \Rightarrow b_2 = \frac{-9.22 \times 10^6}{1.75 \times 10^9} = -0.0053$$

$$\Rightarrow Y = 10.8 + 0.237x - 0.0053x^2$$

(WHEW!)

$$\begin{bmatrix} n & \Sigma x_j & \Sigma x_j^2 \\ \Sigma x_j & \Sigma x_j^2 & \Sigma x_j^3 \\ \Sigma x_j^2 & \Sigma x_j^3 & \Sigma x_j^4 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Sigma Y_j \\ \Sigma Y_j x_j \\ \Sigma Y_j x_j^2 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 0 & 1300 \\ 0 & 1300 & 0 \\ 1300 & 0 & 121420 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 263.5 \\ 308.5 \\ 13418 \end{bmatrix}$$

$$\Delta = 25 \times 1300 \times 121420 - (1300)^3 = 1.750 \times 10^9$$

$$\Delta_{b_0} = (263.5)(1300)(121420) - (1300)^2(13414)$$
$$= 1.890 \times 10^{10} \Rightarrow b_0 = \frac{1.89 \times 10^{10}}{1.75 \times 10^9} = 10.8$$

$$\Delta_{b_1} = 25(308.5)(121420) - (1300)^2(308.5)$$
$$= 4.15 \times 10^8 \Rightarrow b_1 = \frac{4.15 \times 10^8}{1.75 \times 10^9} = 0.237$$

$$\Delta b_2 = 25(1300)(13418) - (263.5)(1300)^2$$
$$= -9.22 \times 10^6 \Rightarrow b_2 = \frac{-9.22 \times 10^6}{1.75 \times 10^9} = -0.0053$$

$$\Rightarrow Y = 10.8 + 0.237x - 0.0053x^2$$

(WHEW!)

Pg 252

10) a)

x	Y		$Y' = a + bX'$
0	12.0		10.48
1	10.5		10.09
2	10.0		9.71
3	8.0		9.33
4	7.0		8.94
5	8.0		8.56
6	7.5		8.18
7	8.5		7.79
8	9.0	$n = 8$	7.41

$\Sigma x_j = 36$; $\Sigma x_j^2 = 204$

$\Sigma Y_i = 80.5$ $\Sigma Y_j^2 = 740.75$

$\Sigma x_j Y_j = 299$

$\Rightarrow \overline{X} = 4$; $\overline{Y} = 8.944$

$\Sigma(x_j - \overline{X})^2 = \Sigma x_j^2 - n\overline{X}^2 = 60$

$\Sigma(Y_j - \overline{Y})^2 = \Sigma Y_j^2 - n\overline{Y}^2 = 20.7222 (= SST)$

$\Sigma(x_j - \overline{X})(Y_j - \overline{Y}) = \Sigma x_j Y_j - n\overline{X}\overline{Y} = -22.9999$

$b = \dfrac{\Sigma(x_j - \overline{x})(y_j - \overline{y})}{\Sigma(x_j - \overline{x})^2} = -0.3833$

$a = \overline{Y} - b\overline{x} = 10.477$

$\Rightarrow Y' = 10.477 - 0.3833X$

$SSR = b^2 \Sigma(x_i - \overline{x})^2 = 8.8166$

$SSD = SST - SSR = 11.9056$

$S_e^2 = \dfrac{SSD}{n-2} = 1.7$

$S_b^2 = \dfrac{S_e^2}{\Sigma(x_j - \overline{x})^2} = 0.28 \Rightarrow S_b = 0.168$

$H_0 : \beta = 0$

$H_1 : \beta \neq 0$

$\alpha = 0.05$

$T = \frac{b}{S_b} = -2.28$

$t_{n-2;\alpha/2} = t_{7;0.025} = -2.365$

$\Rightarrow$ <u>H_0</u> CANNOT BE REJECTED

(SEE PREVIOUS TABLE FOR Y')

$\hat{\sigma}_1^2 = \frac{\Sigma(Y_i - Y_i')^2}{n-2} = 1.70$

b)

Y	Y'	X
12.0	12.200	0
10.5	10.533	1
10.0	9.232	2
8.0	8.300	3
7.0	7.730	4
8.0	7.525	5
7.5	7.690	6
8.5	8.220	7
9.0	9.112	8

$\hat{\sigma}_2^2 = \frac{\Sigma(Y_i - Y_i')^2}{n-3} = 6 = 0.268$

$F = \frac{\hat{\sigma}_1^2 - \hat{\sigma}_2^2}{\hat{\sigma}_2^2} = 5.34$

$H_0 : \beta_2 = 0 \qquad H_1 : \beta_2 \neq 0 \quad ; \alpha = 0.05$

$F_{1;n-3;\alpha} = F_{1,6;0.05} = 5.99$

$\Rightarrow$ ACCEPT H_0

Pg. 252-3

13) $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

$\Sigma X_{1j} = 1.200 \qquad \Sigma X_{1j}^2 = 0.1712$

$\Sigma X_{2j} = 1.38 \times 10^4 ; \Sigma X_{2j}^2 = 1.602 \times 10^7$

$\Sigma X_{1j} X_{2j} = 1.38 \times 10^3$

$\Sigma Y_j = 794.8$

$\Sigma Y_j X_{1j} = 81.14 \quad ; \Sigma Y_j X_{2j} = 9.011 \times 10^5$

$$\begin{bmatrix} n & \Sigma X_{1j} & \Sigma X_{2j} \\ \Sigma X_{1j} & \Sigma X_{1j}^2 & \Sigma X_{1j} X_{2j} \\ \Sigma X_{1j} & \Sigma X_{1j} X_{2j} & \Sigma X_{2j}^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Sigma Y_j \\ \Sigma Y_j X_{1j} \\ \Sigma Y_j X_{2j} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 12.0 & 1.20 & 1.38 \times 10^4 \\ 1.20 & 0.1712 & 1.38 \times 10^3 \\ 1.38 \times 10^4 & 1.38 \times 10^3 & 1.602 \times 10^7 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 794.8 \\ 81.14 \\ 9.011 \times 10^5 \end{bmatrix}$$

$\Delta = (12)(0.1712)(1.602) \times 10^7 + 1.20 \times (1.38 \times 10^3)^2$

$\quad + (1.38 \times 10^3)(1.38 \times 10^4)(1.20)$

$\quad - (12)(1.38 \times 10^3)^2 - (1.38 \times 10^4)^2 0.1712 - (1.2)^2 1.602 \times 10^7$

$\quad = 8.947 \times 10^4$

$\Delta_{b_0} = (794.8)(0.172)(1.602 \times 10^7) + (1.20)(1.38 \times 10^3)(9.011 \times 10^5)$

$\quad + (81.14 \times 1.38 \times 10^3 \times 1.38 \times 10^4) - (794.8)(1.38 \times 10^3)^2$

$\quad - 81.14 \times 1.20 \times 1.602 \times 10^7 - 9.011 \times 10^5 \times 0.1712 \times 1.38 \times 10^4$

$\quad = 1.484 \times 10^7$

$\Delta b_1 = (12.0)(81.14)(1.602 \times 10^7) + 794.8 \times (1.38)^2 \times 10^7$

$\quad + 1.38 \times 10^3 \times 1.20 \times 9.011 \times 10^5 - 12.0 \times 9.011 \times 10^5 \times 1.38 \times 10^4$

$\quad - 1.20 \times 794.8 \times 1.602 \times 10^7 - (1.38)^2 \times 10^7 \times 81.14$

$\quad = 3.036 \times 10^6$

$$\Delta b_2 = (12)(0.1712)(9.011\times10^5) + 1.2\times81.14\times1.38\times10^4$$
$$+ 794.8\times1.2\times1.38\times10^3 - 12\times12\times1.38\times10^3\times81.14$$
$$-(1.2)^2\times9.011\times10^5 - 1.38\times10^4\times0.1712\times794.8$$
$$= -8.014\times10^3$$

$$\Rightarrow b_0 = \frac{\Delta b_0}{\Delta} = \frac{165.9}{8.947\times10^4} = 1.66\times10^2$$
$$b_1 = \frac{\Delta b_1}{\Delta} = \frac{3.036\times10^6}{8.947\times10^4} = 33.9$$
$$b_2 = \frac{\Delta b_2}{\Delta} = \frac{-8.014\times10^3}{8.947\times10^4} = -0.0896$$

AND;

$$Y = 1.66\times10^2 + 33.9X_1 - 0.0896X_2$$

Pg. 253

14) $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$

$\Sigma x_{1j} = 15.30 \qquad \Sigma x_{1j}^2 = 29.85$

$\Sigma x_{2j} = 9.39 \times 10^2 \qquad \Sigma x_{2j}^2 = 9.413 \times 10^4$

$\Sigma x_{1j} x_{2j} = 1.459 \times 10^3$

$\Sigma Y_i = 84.8$

$\Sigma x_{1j} Y_j = 132.27 \quad ; \Sigma x_{2j} Y_j = 8.32 \times 10^3$

$$\begin{bmatrix} n & \Sigma x_{1j} & \Sigma x_{2j} \\ \Sigma x_{1j} & \Sigma x_{1j}^2 & \Sigma x_{1j} x_{2j} \\ \Sigma x_{2j} & \Sigma x_{1j} x_{2j} & \Sigma x_{2j}^2 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \Sigma Y_j \\ \Sigma Y_j x_{1j} \\ \Sigma Y_j x_{2j} \end{bmatrix}$$

$$\begin{bmatrix} 10 & 15.30 & 939 \\ 15.3 & 29.85 & 1459 \\ 939 & 1458.9 & 94130 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 84.6 \\ 132.27 \\ 8320 \end{bmatrix}$$

$\Delta = 3.786 \times 10^5$

$$\begin{bmatrix} 84.6 & 15.30 & 939 \\ 132.27 & 29.85 & 1459 \\ 8320 & 1458.9 & 94130 \end{bmatrix}$$

$\Delta_{b_0} = 8.46 \times 10^5$

$$\Rightarrow b_0 = \frac{\Delta_{b_0}}{\Delta} = \frac{8.46 \times 10^5}{3.786 \times 10^5} = 2.235$$

$$\begin{bmatrix} 10 & 84.6 & 939 \\ 15.3 & 132.27 & 1459 \\ 939 & 8320 & 94130 \end{bmatrix}$$

$\Delta_{b_1} = 8.72 \times 10^4$

$$\Rightarrow b_1 = \frac{8.72 \times 10^4}{3.786 \times 10^5} = 0.23$$

$$\begin{bmatrix} 10 & 15.3 & 84.6 \\ 15.3 & 29.85 & 132.27 \\ 939 & 1458.9 & 8320 \end{bmatrix}$$

$$\Delta_{b_2} = 2.435 \times 10^4 \Rightarrow b_2 = \frac{2.434 \times 10^4}{3.786 \times 10^5} = 0.0643$$

$$\therefore Y = 2.235 + 0.230 x_1 + 0.0643 x_2$$

$x_1 = 2.2$; $x_2 = 90$

$$\Rightarrow Y = 2.235 + 0.230(2.2) + (0.0643)(90)$$
$$\doteq 8.528$$

17)

Y	Z_1	Z_2
78.8	-1	-3
65.1	-1	-1
55.4	-1	1
56.2	-1	3
80.9	0	-3
69.5	0	-1
57.4	0	1
55.2	0	3
85.6	1	-3
71.8	1	-1
60.2	1	1
58.7	1	3

$\Sigma Z_{1j} = \Sigma Z_{2j} = 0$

$\Sigma Z_{1j}^2 = 8$

$\Sigma Z_{2j}^2 = 60$

$\Sigma Z_{1j} Z_{2j} = 0$

$\Sigma Y_j = 794.8$

$\Sigma Y_j Z_{1j} = 20.8$

$\Sigma Y_j Z_{2j} = -259.0$

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 60 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 794.8 \\ 20.8 \\ -259 \end{bmatrix}$$

$$\Rightarrow 12 b_0 = 794.8 \Rightarrow b_0 = 66.23$$

$$8 b_1 = 20.8 \Rightarrow b_1 = 2.6$$

$$60 b_2 = -259 \Rightarrow b_2 = -4.32$$

$$\therefore Y = 66.23 + 2.6 Z_1 - 4.32 Z_2$$

$$= 66.23 + 2.6 \left[\frac{X_1 - 0.10}{0.08} \right] - 4.32 \left[\frac{X_2 - 1150}{50} \right]$$

$$= 162.2 + 32.50 X_1 - 0.0863 X_2$$

PRETTY CLOSE TO #13

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Pg 274

13) $SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (Y_{ij} - \bar{Y})^2$

$= \sum\sum (Y_{ij})^2 - 2\sum\sum \bar{Y} Y_{ij} + \sum\sum \bar{Y}^2$

$= \sum\sum (Y_{ij})^2 - 2\bar{Y}^2 \sum_{i=1}^{k} n_i + \bar{Y}^2 \sum n_i$

$= \sum\sum (Y_{ij})^2 - \bar{Y}^2 \sum_{i=1}^{k} n_i$

BUT $T_i = \sum_{j=1}^{k} Y_{ij}$; $T = \sum_{i=1}^{n_i} T_i = \sum_{i=1}^{k} \sum^{n_i} T_{ij}$

$\Rightarrow SST = \sum_{j=1}^{k} \sum_{i=1}^{n_i} (Y_{ij})^2 - T^2 / \sum_{i=1}^{n_i} n_i$

$SST_r = \sum_{i=1}^{n_i} n_i (\bar{Y}_i - \bar{Y})^2$

$= \sum n_i \bar{Y}_i^2 - 2\sum \bar{Y} \bar{Y}_i n_i + \sum n_i \bar{Y}^2$

$= \sum T_i^2/n_i - 2\bar{Y} \sum T_i + \bar{Y}^2 \sum n_i$

$= \sum T_i^2/n_i - \bar{Y}^2 \sum n_i$

$= \sum T_i^2/n_i - T^2/\sum n_i$

$SSE = \sum_{i=1}^{n_i} \sum_{j=1}^{k} (Y_{ij} - \bar{Y}_i)^2$

$= \sum\sum (Y_{ij})^2 - 2\sum\sum Y_{ij} \bar{Y}_i + \sum\sum (\bar{Y}_i)^2$

$= \sum\sum Y_{ij}^2 - 2\sum_{i=1}^{n_i} \bar{Y}_i T_i + \sum_{i=1}^{n_i} T_i^2/n_i$

$= \sum\sum Y_{ij} - 2\sum T_i^2/n_i + \sum T_i^2/n_i$

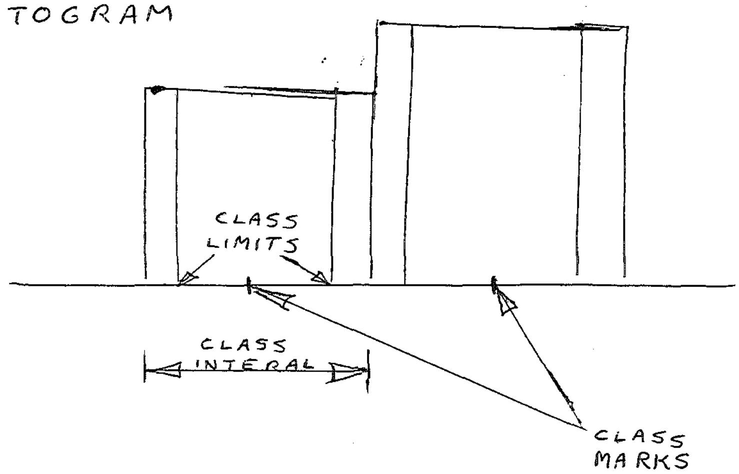
$= \sum\sum Y_{ij}^2 - \sum T_i^2/n_i$

$\therefore SST_r + SSE = \sum_{i=1}^{n_i} \sum_{j=1}^{k} Y_{ij}^2 - \frac{T^2}{\sum n_i} = SST$

1 QUIT

FINAL CRAM AND REFERENCE SHEET
(MEMORIZE)

I) HISTOGRAM



II) TREATMENT OF DATA:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \text{(MEAN)}$$

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \qquad \text{(VARIANCE)}$$

$$s = \sqrt{s^2} \qquad \text{(STANDARD DEVIATION)}$$

GROUPED DATA:

$$x_g = \frac{1}{n} \sum_{i=1}^{k} x_i f_i$$

$$s_g^2 = \frac{1}{n-1} \sum_{i=1}^{k} f_i (x_i - \bar{x}_g)^2$$

CODING:

LET $U_i = \frac{1}{c}(x_i - A)$

$$\Rightarrow \bar{x} = c\bar{U} + A \quad ; \quad S_x = c S_u$$

III) SAMPLING:

A) WITH REPLACEMENT:

$$\mu_{\bar{x}} = \mu \;;\; \sigma_{\bar{x}} = \sigma/\sqrt{n}$$

WITHOUT REPLACEMENT:

$$\mu_{\bar{x}} = \mu \;;\; \sigma^2_{\bar{x}} = \frac{N-n}{N-1}\;\frac{\sigma^2}{n}$$

B) SAMPLING FROM A NORMAL POPULATION

1) $P[\mu < a] = P[t_r > t_{r:\alpha}] = \alpha$

$$\ni\; t = \frac{\bar{x} - a}{s/\sqrt{n}} \qquad (t_{r;1-\alpha} = -t_{r;\alpha})$$

2) $P[s^2 < a^2] = P[\chi^2 < \chi^2_{r:\alpha}] = \alpha$

$$\ni\; \chi^2 = \frac{(n-1)a^2}{\sigma^2}$$

3) $P\left[\frac{s_1^2}{s_2^2} > a\right] = P[F_{r_1, r_2:\alpha} > a] = \alpha$

$$F_{1-\alpha; r_1, r_2} = \frac{1}{F_{\alpha; r_2, r_1}}$$

IV) CONFIDENCE INTERVALS

FOR $\bar{x}$

σ KNOWN: $P\left[\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$

σ UNKNOWN: $P\left[\bar{x} - t_{n-1;\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{n-1;\alpha/2}\frac{s}{\sqrt{n}}\right] = 1-\alpha$

FOR σ^2

$n<30$; $P\left[\frac{(n-1)s^2}{\chi^2_{n-1;\alpha/2}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{n-1;1-\alpha/2}}\right] = 1-\alpha$

$n \ge 30$; $P\left[\frac{s}{1+Z_{\alpha/2}/\sqrt{2n}} \le \sigma \le \frac{s}{1-Z_{\alpha/2}/\sqrt{2n}}\right] = 1-\alpha$

V) TEST OF HYPOTHESIS

A) ONE MEAN; $H_0: \mu = \mu_0$

1) $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ (σ KNOWN)

2) $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$ (σ UNKNOWN)

$\gamma = n - 1$

B) TWO MEANS; $H_0: \mu_1 - \mu_2 = \delta$

1) σ_1 AND σ_2 KNOWN; INDEP. POPULATIONS

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

2) σ_1 AND σ_2 UNKNOWN, BUT $\sigma_1 = \sigma_2$

$$t = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{s\sqrt{1/n_1 + 1/n_2}}$$

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$\gamma = n_1 + n_2 - 2$$

3) σ_1 & σ_2 UNKNOWN, $\sigma_1 \neq \sigma_2$

$$t = \frac{\bar{X}_1 - \bar{X}_2 - \delta}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

$$\gamma = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$$

4) σ_1 & σ_2 UNKNOWN, $\sigma_1 \neq \sigma_2$; DEPENDENT POP.

FIND $\bar{X}_{d_i} \ni d_i = X_{1i} - X_{2i}$

$$t = \frac{\bar{X}_{d_i} - \delta}{S_{d_i}/\sqrt{n}}$$

C) ONE VARIANCE, $H_0: \sigma^2 = \sigma_0^2$

1) $n \leq 31$: $\chi^2 = \dfrac{1}{\sigma_0^2}(n-1)s^2$; $\gamma = n - 1$

2) $n > 31$: $Z = \dfrac{s - \sigma_0}{\sigma_0/\sqrt{2n}}$

D) TWO VARIANCES; $H_0: \sigma_1^2 = \sigma_2^2$

H_1	FIND	REJECT H_0 IF
$\sigma_1^2 < \sigma_2^2$	$F = s_2^2/s_1^2$	$F > F_{\alpha;\, n_2 - 1,\, n_1 - 1}$
$\sigma_1^2 > \sigma_2^2$	$F = s_1^2/s_2^2$	$F > F_{\alpha;\, n_1 - 1;\, n_2 - 1}$
$\sigma_1^2 \neq \sigma_2^2$	$F = s_M^2/s_m^2 > 1$	$F > F_{\alpha/2}(n_M - 1,\, n_m - 1)$

VI) PROBABILITY INFERENCE

A) FOR LARGE n; $H_o: p = p_o$

$$Z = \frac{x - np_o}{\sqrt{np_o(1-p_o)}}$$

B) FOR LARGE n_1, n_2; $H_o: p_1 = p_2$

$$Z = \frac{x_1/n - x_2/n}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}} \quad ; \quad \hat{p} = \frac{\sum \text{SUCCESSES}}{n}$$

C) FOR $H_o: P_1 = P_2 = P_3 = \ldots = P_j = \ldots = P_K = P_o$

ALWAYS USE AN UPPER TAIL TEST

SUCCESS	$\boxed{e_{11}}$ f_{11}	$\boxed{e_{12}}$ f_{12}	. .	. .	.	. .	$\boxed{e_{1k}}$ f_{1K}	Σf_{1j} b
FAILURE	$\boxed{e_{21}}$ f_{21}	$\boxed{e_{22}}$ f_{22}	. .	.	.	. .	f_{2K}	Σf_{2j}
	Σf_{i1} a	Σf_{i2}					Σf_{ij}	n TRIALS

$\cancel{e_{ij} = n_j \hat{p}}$ $\quad$ $\cancel{e_{ij} = \frac{1}{n} \Sigma f_{ij} f_i}$ $\quad$ $e_{11} = \frac{ab}{n}$

COMPUTE $\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$

$e_{1j} = n_j \hat{p} = n_j \frac{x}{n}$

$e_{2j} = \cancel{\Sigma}\, n_j (1 - \frac{x}{n})$

REJECT H_o FOR $\chi^2 > \chi^2_{K-1;\alpha}$

D) CONTINGENCY TABLES

COMPUTE $\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$

REJECT H_o IF $\chi^2 > \chi^2_{(r-1)(K-1);\alpha}$

VII) SHORT CUT INFERENCE

A) SIGN TEST; $H_0: \mu = \mu_0$

$+ \Rightarrow \; > \mu_0 \; ; \; - \Rightarrow \; < \mu_0 \Rightarrow H_0 : p = \frac{1}{2}$

B) RANK SUM TEST: H_0: IDENTICAL POPULATIONS
(MANN - WHITNEY)

ARRANGE DATA IN ASCENDING ORDER

ASSIGN EACH $1, 2, 3, \ldots, n_1 + n_2$

$\mu_0 = \frac{n_1 n_2}{2} \; ; \; \sigma_U^2 = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}$

$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$

$Z = \frac{U - \mu_\mu}{\sigma_\mu}$

C) KRUSKAL - WALLIS TEST - H_0: IDENTICAL POPULATIONS

ARRANGE DATA AS ABOVE

$H = \chi^2 = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{R_i}{n_i} - 3(n+1)$; USE $\gamma = k - 1$

UPPER TAIL TEST

D) RUN TEST

n_1 SUCCESSES (S) ; n_2 FAILURES F

μ = # OF RUNS

$\mu_\mu = \frac{2 n_1 n_2}{n_1 + n_2} + 1$

$\sigma_\mu^2 = \frac{2 n_1 n_2 (2 n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)}$; $Z = \frac{U - \mu_\mu}{\sigma_\mu}$

TREND $\rightarrow$ LTT

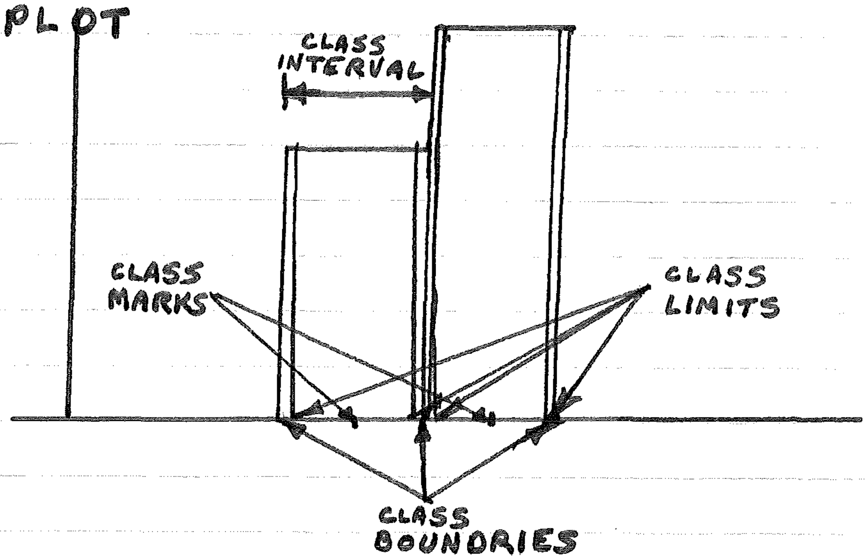
CYCLE $\rightarrow$ UTT

STATISTICAL METHODS I

I) TREATMENT OF DATA

A) HISTOGRAM GENERATION

1) TALLY DATA INTO A FREQUENCY DISTRIBUTION
2) PLOT



B) UNGROUPED DATA

1) MEAN: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$

2) VARIANCE: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \bar{X}^2\right]$

3) STANDARD DEVIATION: $S = \sqrt{S^2}$

4) CODING:

LET $U_i = \frac{1}{C}(x_i - A)$

THEN $\bar{X} = C\bar{U} + A$; $S_x = C S_U$

C) GROUPED DATA

1) MEAN: $\overline{X_g} = \frac{1}{n} \sum_{i=1}^{k} x_i f_i$

2) VARIANCE: $S_g^2 = \frac{1}{n-1} \sum_{i=1}^{k} f_i (x_i - \bar{x}_g)^2$

$= \frac{1}{n-1}\left[\sum_{i=1}^{k} f_i x_i^2 - n x_g^2\right]$

II) SAMPLING DISTRIBUTIONS

A) VARIANCES:

1) POPULATION VARIANCE: σ^2

2) SAMPLE VARIANCE: s^2

B) SAMPLING WITH σ^2 & μ KNOWN

1) SAMPLING WITH REPLACEMENT

$$\mu_{\bar{x}} = \mu$$
$$\sigma_{\bar{x}} = \sigma/\sqrt{n}$$

2) WITHOUT REPLACEMENT

$$\mu_{\bar{x}} = \mu$$
$$\sigma_{\bar{x}}^2 = \frac{N-n}{N-1}\,\frac{\sigma^2}{n}$$

3) $P[\bar{x} < a] = P[z < \frac{a-\mu}{\sigma}]$

C) SAMPLING WITH ONLY μ KNOWN
(SAMPLING FROM A NORMAL POPULATION) $(\gamma = n-1)$

THEN $P[\mu < a] = P[t_\gamma > t_{\gamma:\alpha} = \frac{\bar{x}-a}{s/\sqrt{n}}] = \alpha$

D) VARIANCE DISTRIBUTION (σ^2 & μ KNOWN FROM A NORMAL POPULATION)

$$P[s^2 < a^2] = P[\chi_\gamma^2 < \frac{(n-1)a^2}{\sigma^2} = \chi^2_{\gamma;\alpha}] = \alpha$$

E) VARIANCE RELATIONSHIP (n_1 & n_2 SAMPLES FROM A NORMAL POPULATION)

$$P[\frac{s_1^2}{s_2^2} > a] = P[F_{\gamma_1,\gamma_2;\alpha} > a] = \alpha$$

F) PARAMETER RELATIONSHIPS

1) $t_{\gamma;1-\alpha} = -t_{\gamma;\gamma}$

2) $F_{1-\alpha}(\gamma_1, \gamma_2) = 1/F_\alpha(v_2, v_1)$

III) ERROR TYPES

A) $\alpha = P[\text{TYPE I ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu = \mu_0]$

$= P[\bar{X} \text{ EXCEEDS BOUNDRY(S) GIVEN } \mu = \mu_0]$

$\beta = P[\text{TYPE II ERROR}]$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN ALTERNATIVE}]$

B) O.C. CURVE

$g(\mu) = P[\text{TYPE II ERROR GIVEN } \mu]$

IV) CONFIDENCE INTERVALS

A) FOR $\overline{X}$ (α = LEVEL OF SIGNIFICANCE)

1) σ^2 KNOWN:

$$P\left[\overline{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

2) SAMPLING FROM A $N(\mu, \sigma^2)$ POP WITH σ^2 UNKNOWN

$$P\left[\overline{X} - t_{n-1;\alpha/2}\frac{s}{\sqrt{n}} < \mu < \overline{X} + t_{n-1;\alpha/2}\frac{s}{\sqrt{n}}\right] = 1-\alpha$$

B) FOR σ^2 AND σ

1) SAMPLING FROM A $N(\mu, \sigma^2)$ POP; $n \leq 31$

$$P\left[\frac{(n-1)s^2}{\chi^2_{n-1;\alpha/2}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1;1-\alpha/2}}\right] = 1-\alpha$$

2) FOR $n > 31$

$$P\left[\frac{s}{1 + Z_{\alpha/2}/\sqrt{2n}} \leq \sigma \leq \frac{s}{1 - Z_{\alpha/2}/\sqrt{2n}}\right] = 1-\alpha$$

V) TEST OF HYPOTHESES (GIVEN α)

A) FOR ONE MEAN: $H_0: \mu = \mu_0$

1) σ^2 KNOWN

FIND $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

H_1	REJECT H_0 IF
$\mu < \mu_0$	$Z < -Z_\alpha$
$\mu > \mu_0$	$Z > Z_\alpha$
$\mu \neq \mu_0$	$Z < -Z_{\alpha/2}$ OR $Z > Z_{\alpha/2}$

2) σ^2 UNKNOWN (SAMPLE FROM A $N(\mu, \sigma^2)$ POP.)

FIND $t = \bar{X} - \mu_0 / s/\sqrt{n}$; $(\nu = n-1)$

H_1	REJECT H_0 IF
$\mu < \mu_0$	$t < -t_\alpha$
$\mu > \mu_0$	$t > t_\alpha$
$\mu \neq \mu_0$	$t < -t_{\alpha/2}$ OR $t > t_{\alpha/2}$

B) FOR TWO MEANS: $H_0: \mu_1 - \mu_2 = \delta$

1) σ_1^2 AND σ_2^2 ARE KNOWN

SAMPLES FROM INDEPENDENT POPULATIONS

FIND $Z = \dfrac{\bar{X}_1 - \bar{X}_2 - \delta}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$

H_1	REJECT H_0 IF
$\mu_1 - \mu_2 < \delta$	$Z < -Z_\alpha$
$\mu_1 - \mu_2 > \delta$	$Z > Z_\alpha$
$\mu_1 - \mu_2 \neq 0$	$Z < -Z_{\alpha/2}$ OR $Z > Z_{\alpha/2}$

2) σ_1^2 AND σ_2^2 UNKNOWN $\Rightarrow \sigma_1^2 = \sigma_2^2$

SAMPLE FROM INDEPENDENT $N(\mu, \sigma^2)$ POP.

FIND $t = \dfrac{\bar{X}_1 - \bar{X}_2 - \delta}{s\sqrt{1/n_1 + 1/n_2}}$

WHERE $s^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$

H_1	REJECT H_0 IF
$\mu_1 - \mu_2 < \delta$	$t < -t_\alpha$
$\mu_1 - \mu_2 > \delta$	$t > t_\alpha$
$\mu_1 - \mu_2 \neq \delta$	$t < -t_{\alpha/2}$ OR $t > t_{\alpha/2}$

3) σ_1^2 AND σ_2^2 UNKNOWN $\Rightarrow \sigma_1^2 \neq \sigma_2^2$

SAMPLE FROM INDEPENDENT $N(\mu, \sigma^2)$ POP.

FIND $t' = \dfrac{\bar{X}_1 - \bar{X}_2 - \delta}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$

AND $\nu = \dfrac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2/n_2)^2}{n_2 - 1}}$

(SAME TEST AS ABOVE)

4) σ_1^2 AND σ_2^2 UNKNOWN $\Rightarrow \sigma_1^2 \neq \sigma_2^2$

SAMPLE FROM DEPENDENT $N(\mu, \sigma^2)$ POP.

FIND $\bar{X}_{di} \ni d_i = x_{1i} - x_{2i}$

AND $t = \dfrac{\bar{X}_{di} - \delta}{s_{di}/\sqrt{n}}$

(SAME TEST AS ABOVE)

C) FOR ONE VARIANCE : $H_0 : \sigma^2 = \sigma_0^2$

1) $N(\mu, \sigma^2)$ POPULATION ; $n \leq 31$

FIND $\chi^2 = \frac{1}{\sigma_0^2}(n-1)S^2$

H_1	REJECT H_0 IF	$(\nu = n-1)$
$\sigma^2 < \sigma_0^2$	$\chi^2 < \chi^2_{1-\alpha}$	
$\sigma^2 > \sigma_0^2$	$\chi^2 > \chi^2_{\alpha}$	
$\sigma^2 \neq \sigma_0^2$	$\chi^2 < \chi^2_{1-\frac{\alpha}{2}}$ OR $\chi^2 > \chi^2_{\alpha/2}$	

2) $N(\mu, \sigma^2)$ FROM $N(\mu, \sigma^2)$ POP. $n > 31$

FIND $Z = \dfrac{S - \sigma_0}{\sigma_0 / \sqrt{2n}}$

D) TWO VARIANCES: $H_0 : \sigma_1^2 = \sigma_2^2$

$\cancel{\nu = n_1 + n_2 - 2}$

H_1	FIND	REJECT H_0 IF
$\sigma_1^2 < \sigma_2^2$	$F = S_2^2 / S_1^2$	$F > F_\alpha(n_2 - 1, n_1 - 1)$
$\sigma_1^2 > \sigma_2^2$	$F = S_1^2 / S_2^2$	$F > F_\alpha(n_1 - 1, n_2 - 1)$
$\sigma_1^2 > \sigma_2^2$	$F = S_M^2 / S_m^2 > 1$	$F > F_{\alpha/2}(n_M - 1, n_m - 1)$

$$\Rightarrow F_{\nu_1, \nu_2 ; 1-\alpha} = \frac{1}{F_{\nu_2, \nu_1 ; \alpha}}$$

PROPORTION INFERENCE

FOR $H_0: p = p_0$ WITH LARGE n:

$$Z = \frac{x - np_0}{\sqrt{np_0(1-p_0)}}$$

FOR $H_0: p_1 = p_2$

$$Z = \frac{x_1/n - x_2/n_2}{\sqrt{\hat{p}(1-\hat{p})(1/n_1 + 1/n_2)}}$$

FOR $H_0: P_1 = P_2 = P_3 = \ldots = P_j = \ldots = P_K = P_0$

ALWAYS USE AN UPPER TAIL TEST

SET UP TABLE:

SUC.	f_{11}	f_{12}	f_{13}	$\ldots$	f_{1j}	$\ldots$	f_{1K}	$\Rightarrow \Sigma f_{1j}$
FAIL	f_{21}	f_{22}	$\ldots$	$\ldots$	f_{2j}	$\ldots$	f_{2K}	Σf_{2j}
	Σf_{i1}	Σf_{i}						n = TOTAL TRIALS

$e_{ij} = n_j \hat{p} = n_j \frac{x}{n}$; $e_{2j} = n_j (1 - \frac{x}{n})$

COMPUTE $\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$

REJECT H_0 IF $\chi^2 > \chi^2_{K-1;\alpha}$

CONTINGINCY TABLES EXTEND THIS NOTION

$$\chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

REJECT H_0 IF $\chi^2 > \chi^2_{(r-1)(K-1)}$

SHORT-CUT INFERENCE

SIGN TEST: $H_0: \mu = \mu_0$ $\quad$ n TRIALS $\Rightarrow H_0: p = 1/2$

$+ \rightarrow > \mu_0$; $- \rightarrow < \mu_0$,

RANK-SUM TEST: $H_0: \mu_1 = \mu_2$ IDEN. POP. $\quad H_1: \mu_1 \neq \mu_2$ DIFFER. MEANS (MANN WHITNEY)

ARRANGE DATA IN ASCENDING ORDER

ASSIGN EACH $1, 2, 3, \ldots, n_1+n_2$

$$\Rightarrow \mu_U = \frac{n_1 n_2}{2} ; \quad \sigma_\mu^2 = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}$$

$$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$$

$$Z = \frac{U - \mu_U}{\sigma_U}$$

KRUSKAL-WALLIS TEST $\quad H_0$: IDENTICAL POP. $\quad H_1$: DIFFERENT MEANS

ARRANGE DATA AS ABOVE

$$H = \chi^2_{k-1} = \frac{12}{n(n+1)} \sum_{i=1}^{k} \frac{R_i^2}{n_i} - 3(n+1)$$

USE UPPER TAIL TEST

(RUN) RANDOMNESS TEST

n_1 SUCCESSES (S), n_2 FAILURES (F)

μ = # OF RUNS

$$\mu_\mu = \frac{2 n_1 n_2}{n_1 + n_2} + 1$$

$$\sigma_\mu^2 = \frac{2 n_1 n_2 (2 n_1 n_2 - n_1 - n_2)}{(n_1 + n_2)^2 (n_1 + n_2 - 1)} ; \quad Z = \frac{\mu - \mu_\mu}{\sigma_\mu}$$

TREND $\rightarrow$ LOWER TAIL

CYCLE $\rightarrow$ UPPER TAIL

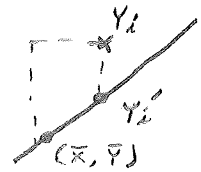
CURVE FITTING

$Y_i = \alpha + \beta x_i + e_i$

$b = \dfrac{\Sigma x_i Y_i - n\bar{x}\bar{Y}}{\Sigma x_i^2 - n\bar{x}^2} \Rightarrow E(b) = \beta$

$a = \bar{Y} - b\bar{x} \qquad \Rightarrow E(a) = \alpha$

$SST = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$



$SSD = \Sigma \, (Y_i - Y_i')^2$

$SSR = \Sigma (Y_i' - Y)^2$

$SST = SSD + SSR$

$SSR = b^2 \Sigma (x_i - \bar{x})^2$

$S_e^2 = SSD/n-2 \quad ; \quad E[S_e^2] = \sigma^2$

$S_a^2 = \dfrac{S_e^2 \Sigma x_i}{n \Sigma (x_i - \bar{x})^2} \quad ; \quad S_b^2 = \dfrac{S_e^2}{\Sigma (x_i - \bar{x})}$

$T = \dfrac{a - \alpha}{S_a} \quad ; \quad T = \dfrac{b - \beta}{S_b}$

CONFIDENCE INTERVALS

$a \pm t_{n-2;\alpha/2} S_a (\alpha) ; \quad b \pm t_{n-2;\alpha/2} S_b (\beta)$

FOR $H_0 : \beta = b_0$

$H_1 : \beta > b_0 \Rightarrow$ REJECT H_0 IF $T > t_{n-2;\alpha}$

$H_1 : \beta < b_0 \Rightarrow$ " " " $T < -t_{n-2;\alpha}$

$H_1 : \beta \neq b_0 \Rightarrow$ " " " $T > t_{n-2;\alpha/2}$

IF x_0 IS A GIVEN ~~VALUE OF X~~: OR $T < -t_{n-1;\alpha/2}$

$$T = \frac{a + bx_0 - (\alpha + \beta x_0)}{S_e \sqrt{1/n + \frac{(x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}} \quad ; \; n-2 = \nu$$

CONF. INTERVAL $(1-\alpha)$ FOR $E[Y]$

$$a + bx_0 \pm t_{n-2;\alpha/2} S_e \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}$$

FOR Y (LIMIT OF PERDICTION)

$$T = \frac{Y - a - bx_0}{S_e \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}}$$

CONF. INTERVAL: $a + bx_0 \pm t_{n-2;\alpha/2} S_e \sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}$

MA 506 - Part 1

(29) 1. The data {405, 402, 403, 400, 40[illegible]} are a random sample from a population. Compute the sample <u>mean</u>, <u>median</u>, <u>range</u>, <u>variance</u>, and <u>standard deviation</u>.

5

(25) 2. The numbers $-1, 0, 1, 2$ form a <u>population</u>.

(a) Compute the <u>population</u> mean μ and variance σ^2.

(b) Suppose a random sample of size 3 is taken <u>without</u> replacement from the population. Compute the probability function of $\overline{X}$, the sample mean random variable. (Hint: list the random samples without taking order into account. There will be only 4 different samples then.)

(c) Using the distribution of $\overline{X}$ in (b), compute the mean $\mu_{\overline{X}}$ and variance $\sigma^2_{\overline{X}}$ of the distribution of $\overline{X}$. Then verify that $\mu_{\overline{X}} = \mu$ and $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n} \frac{N-n}{N-1}$, where N is the population size and n is the sample size.

(10) [illegible] ... is taken [illegible]
a normal population with unknown mean
μ and known variance $\sigma^2 = 4$. The
mean $\bar{X}$ is then used as an estimate of μ.
Compute the probability $\bar{X}$ differs from μ
by more than 0.5 units.

(15) 4. A manufacturing process yields 10 %
defective items. If a random sample of
100 items is taken from the process,
use the normal approximation to the
binomial distribution with the "$\frac{1}{2}$ correction
factor" to compute the probability that
there are more than 13 defective items
in the sample.

(16) 5. A random sample of n is taken
from a $N(\mu, \sigma^2)$ population.
Specify completely the distribution of

(a) $\bar{X}$, (b) $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$, (c) $\dfrac{(n-1)S^2}{\sigma^2}$, (d) [illegible]

where $\bar{X}$ and S^2 are the sample mean
and sample variance random variables
respectively.

BOB MARKS

1)

X_i	$U_i = X_i - 400$	U_i^2	X_i^2
400	0	0	
401	1	1	
402	2	4	
403	3	9	
405	5	25	
2011	$11 \Rightarrow \bar{U} = \frac{11}{5}$	39	

$$\bar{U} = \frac{1}{n} \sum U_i = \frac{11}{5} = 2.2$$

a) $\bar{X} = \bar{U} + 400 = 402.2 \Rightarrow$ SAMPLE MEAN

b) MEDIAN $= 402$

c) RANGE $= X_{MAX} - X_{MIN} = 405 - 400 = 5$

d) $S_{\bar{x}}^2 = \frac{1}{n-1}\left[\sum X_i^2 - n\bar{X}^2\right]$

$$S_{\bar{U}}^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} U_i^2 - n\bar{U}^2\right]$$

$$= \frac{1}{n-1}\left[39 - 5\left(\frac{11}{5}\right)^2\right]$$

$$= \frac{1}{4}\left[39 - \frac{121}{5}\right]$$

$$= \frac{74}{20}$$

$$= \frac{37}{10}$$

$$= 3.7 = S_{\bar{x}}^2$$

e) $S_{\bar{x}} = \sqrt{S_{\bar{x}}^2} = 1.92$

$400 \times 400 = 160000$

$5 \overline{)11.0} = 2.2$

$39 \times 5 = 195$

$5\left(\frac{11^2}{5^2}\right)$

$\frac{195}{5} - \frac{121}{5} = \frac{74}{5}$

$\frac{74}{4}$

3) $n = 25$

$\mu = ?$

$\sigma^2 = 4$

FIND $P[|\bar{x} - \mu| > 0.5] = 1 - P[|\bar{x} - \mu| < 0.5]$

$= 1 - P[-0.5 < \bar{x} - \mu < 0.5]$

$= 1 - P\left[\frac{-0.5}{2} < \frac{\bar{x} - \mu}{\sigma} < \frac{0.5}{2}\right]$

$= 1 - P[-0.25 < z < 0.25]$

ANS.

$= 1 - \{P[z < 0.25] - P[z < -0.25]\}$

$= 1 - P[z < 0.25] + [1 - P[z < 0.25]]$

$= 2[1 - P[z < 0.25]]$

4) $n = 100$; $p = 0.10$

$\mu = np = (100)(0.10) = 10$

$\sigma^2 = np(1-p) = \mu(1-p) = (10)(.9) = 9$

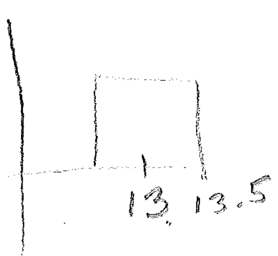
$\Rightarrow \sigma = 3$

$P[n_{def} > 13] = P\left[\frac{n_{def} - np}{\sqrt{np(1-p)}} > \frac{13.5 - np}{\sqrt{np(1-p)}}\right]$

$\approx P\left[z > \frac{13.5\ \mu}{\sigma}\right]$

$= P\left[z > \frac{13.5 - 10}{3}\right]$

$= P\left[z > \frac{3.5}{3}\right] \Rightarrow z = N(0,1)$



$P(z > 1.17)$

MA 506 – Test 1 – Answers

1.

x_i	$u_i = x_i - 400$	u_i^2
405	5	25
402	2	4
403	3	9
400	0	0
401	1	1
	11	39

$\bar{u} = \frac{11}{5} = 2.2$

$\bar{x} = \bar{u} + 400 = 401.2$

$s_u^2 = \dfrac{\sum u_i^2 - \frac{(\sum u_i)^2}{5}}{4}$

$s_u^2 = \dfrac{39 - \frac{121}{5}}{4} =$

$m = 402$

$R = 405 - 400 = 5$

$s_x^2 = s_u^2 = \frac{74}{20} = 3.7$

$s_x = \sqrt{3.7} = 1.9$

2. (a) $\mu = E(X) = (-1)\frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = \frac{1}{2}$

$E(X^2) = 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{6}{4}$

$\sigma^2 = E(X^2) - \mu^2 = \frac{6}{4} - \left(\frac{1}{2}\right)^2 = \frac{5}{4}$

(b)

sample	$\bar{X}$	$p_{\bar{X}}(\bar{x})$
$-1, 0, 1$	0	$\frac{1}{4}$
$-1, 0, 2$	$\frac{1}{3}$	$\frac{1}{4}$
$-1, 1, 2$	$\frac{2}{3}$	$\frac{1}{4}$
$0, 1, 2$	1	$\frac{1}{4}$

$\mu_{\bar{X}} = E(\bar{X})$

$= 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4}$

$+ 1 \cdot \frac{1}{4} = \frac{6}{12} = \frac{1}{2}$

$E(\bar{X}^2) = 0^2 \cdot \frac{1}{4} + \left(\frac{1}{3}\right)^2 \cdot \frac{1}{4} + \left(\frac{2}{3}\right)^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{4} = \frac{14}{36}$

$\sigma_{\bar{X}}^2 = E(\bar{X}^2) - \mu_{\bar{X}}^2 = \frac{14}{36} - \left(\frac{1}{2}\right)^2 = \frac{5}{36}$

Also, $\sigma_{\bar{X}}^2 = \dfrac{\sigma^2}{n} \, \dfrac{N-n}{N-1} = \dfrac{\left(\frac{5}{4}\right)}{3} \, \dfrac{4-3}{4-1} = \dfrac{5}{36}$

3. $P\left(|\bar{X} - \mu| > \frac{1}{2} \right) = P\left(|Z| > \frac{\frac{1}{2}\sqrt{25}}{2} \right)$

$= P(|Z| > 1.25) = P(Z > 1.25) + P(Z < -1.25)$

$= 2\left[1 - P(Z < 1.25)\right] = 2(.1056) = .211$

4. $P(X_B > 13) \approx P(X_N \geq 13.5)$

$= P\left(Z \geq \frac{13.5 - (.10)(100)}{\sqrt{(100)(.1)(.9)}} \right)$

$= P(Z \geq 1.17)$

5 (a) $N\left(\mu, \frac{\sigma^2}{n}\right)$

(b) $N(0,1)$

(c) χ^2_{n-1}

(d) T_{n-1}

3. $P\left(|\bar{X}-\mu| > \frac{1}{2} \right) = P\left(|z| > \frac{\frac{1}{2}\sqrt{25}}{2} \right)$

$= P\left(|z| > 1.25 \right) = P(z > 1.25) - P(z < -1.25)$

$= 2\left[1 - P(z < 1.25) \right] = 2(.1056) = .211$

4. $P\left(X_B > 13 \right) \approx P\left(X_N \geq 13.5 \right)$

$= P\left(z \geq \frac{13.5 - (.10)(100)}{\sqrt{(100)(.1)(.9)}} \right)$

$= P\left(z \geq 1.17 \right)$

5 (a) $N\left(\mu, \frac{\sigma^2}{n} \right)$

(b) $N(0,1)$

(c) χ^2_{n-1}

(d) T_{n-1}

Test 2 — MATH 5[illegible]0 — Open Book

(3) 1. A random sample of 16 from a $N(\mu, \sigma^2)$ population yielded a mean $\overline{X} = 40$ and standard deviation $S = 2$.

(a) Give unbiased estimates of μ and σ^2.

(b) Give 95% confidence intervals for both μ and σ^2.

(c) Test the hypothesis $H_0: \mu = 39$ against $H_1: \mu > 39$ using $\alpha = .05$

(d) Test the hypothesis $H_0: \sigma^2 = 3$ against $H_1: \sigma^2 > 3$.

(2) 2. A random sample of 10 was taken from a $N(\mu_1, \sigma_1^2)$ population and yielded a mean $\overline{X}_1 = 40$ and standard deviation $S_1 = 2$. A random sample of 8 was taken from a second independent $N(\mu_2, \sigma_2^2)$ population and yielded a mean $\overline{X}_2 = 42$ and standard deviation $S_2 = 3$.

(a) Give a 95% confidence interval for $\mu_1 - \mu_2$ assuming $\sigma_1^2 = \sigma_2^2$.

(b) Test $H_0: \mu_1 = \mu_2$ against $H_1: \mu_1 \neq \mu_2$ using $\alpha = .01$ and assuming $\sigma_1^2 = \sigma_2^2$.

(c) Test $H_0: \sigma_1^2 = \sigma_2^2$ against $H_1: \sigma_1^2 \neq \sigma_2^2$ using $\alpha = .02$.

(25) 3. A random sample of 4 girls from the set of girls at I.S.U. were put on a diet and the following data were obtained:

Girl	Weight before diet	Weight after diet
1	140	130
2	120	115
3	100	100
4	110	100

Using $\alpha = .05$ test the null hypothesis that the diet "on the average" does not help girls at I.S.U. lose weight against the alternative it does.

(20) 4. A random sample of 9 is taken from a $N(\mu, \sigma^2)$ population where it is known that $\sigma^2 = 4$. We want to test $H_0: \mu = 5$ against $H_1: \mu < 5$. We shall reject H_0 if the sample mean is less than 4 $(\bar{X} < 4)$.

(a) Compute the probability of Type I error.

(b) Compute the probability of Type II error when $\mu = 3.5$.

Answers for Test 2 - MA 360

1. (a) $\bar{X} = 40$, $S^2 = 4$

(b) $\bar{X} \pm t_{15;.025}\, S/\sqrt{n} = 40 \pm 2.131 \dfrac{(2)}{\sqrt{16}}$

$= 40 \pm 1.066 = \begin{cases} 41.066 \\ 38.934 \end{cases}$

(c) $t = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} = \dfrac{40 - 39}{2/\sqrt{16}} = 2$

$t_{15;.05} = 1.753$ accept H_0

(d) $L.L. = \dfrac{(n-1)S^2}{\chi^2_{15;.025}} = \dfrac{(15)(4)}{27.488} = 2.18$

$U.L. = \dfrac{(15)(4)}{6.262} = 9.58$

(d) $\chi^2 = \dfrac{(n-1)S^2}{\sigma_0^2} = \dfrac{(15)(4)}{3} = 20$

$\chi^2_{15;.01} = 30.578 \quad \chi^2_{15;.05} = 25$

accept H_0

2. $S^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \dfrac{(9)(4)+(7)(9)}{16} = 6.2$

$S = \sqrt{6.2} = 2.48$

(a) $\bar{X}_1 - \bar{X}_2 \pm t_{16;.025}\, S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$

$= -2 \pm 2.12\,(2.48)\sqrt{\frac{1}{10}+\frac{1}{8}} = -2 \pm 2.5 = \begin{cases} .5 \\ -4.5 \end{cases}$

2.(b) $t = \dfrac{\bar{x}_1 - \bar{x}_2}{s\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \dfrac{[illegible]}{(2.48)\sqrt{\frac{1}{10}+\frac{1}{8}}} = -1.[illegible]$

$t_{16;.005} = 2.921$. accept H_0 since $t > -t_{16;.005}$

(c) $F = \dfrac{s_2^2}{s_1^2} = \dfrac{9}{4} = 2.25$

$F_{7,9;.01} = 5.61$ accept H_0

3.

x_{1i}	x_{2i}	d_i	d_i^2
150	140	10	100
120	115	5	25
100	100	0	0
110	100	10	100
		25	225

$\bar{d} = \dfrac{25}{4} = 6.25$

$s_d^2 = \dfrac{n\sum d_i^2 - (\sum d_i)^2}{n(n-1)} = \dfrac{(4)(225)-(25)^2}{(4)(3)} = 23$

$s_d = 4.8$

$t = \dfrac{\bar{d}-0}{s_d/\sqrt{n}} = \dfrac{6.25}{4.8/2} = 2.6$

$H_0: \mu_d = 0$ $\quad t_{3;.05} = 2.353$ reject H_0

$H_1: \mu_d > 0$

4.(a) $\alpha = P\left(Z < \dfrac{4-5}{2/3}\right) = P\left(Z < -\frac{3}{2}\right) = .0668$

(b) $\beta = P\left(Z > \dfrac{4-3.5}{2/3}\right) = P\left(Z > \frac{3}{4}\right) = .2266$

Test 3 — MA 303

(25) 1. A random sample of 200 college students were classified according to class status and drinking habits. The data are given below:

	freshman	sophomore	juniors	seniors
Drinkers	30	30	20	20
Non-drinkers	50	20	20	10

Test the null hypothesis that class status and drinking habits are independent against the alternative they are not using $\alpha = .05$.

(20) 2. In a random sample of 1000 homes in a certain city, it is found that 318 have color television sets. Using $\alpha = .10$ test the null hypothesis that $\frac{1}{3}$ of the homes in the city have color television against the alternative less than $\frac{1}{3}$ have color television.

(7) 3. Given are pairs of data:

x	-2	-1	0	1	2	3
y	-1	-1	0	0	0	2

(a) Compute the least squares line (x the independent variable).

(b) Assuming all the regression assumptions hold and $E(Y) = \alpha + \beta x$ give a 95% confidence interval for β.

(15) 4. The per nicotine content of random samples of two brands of cigarettes, measured in milligrams is found to be as follows:

Brand A	22.1	27.1	26.3	25.7	24.8	25.7	26.1	22.[illegible]
Brand B	27.2	28.6	23.1	22.5	27.0	26.2	24.0	[illegible]

Use the Mann - Whitney U test with $\alpha = .05$ to ~~[illegible]~~ test the hypothesis that the average nicotine contents of the two brands are equal against the alternative they are unequal.

BOB

82

1) $n=16$ $N(\mu,\sigma^2)$.
$\bar{X}=40$
$S=2$

a) $E[\bar{X}]=\mu$ (ie $\bar{X}$ IS UNBIASED ESTIMATE OF μ)
$E[S]=\sigma \Rightarrow [E[S]]^2=\sigma^2$

(4) $(E[S])^2$ IS UNBIASED ESTIMATE OF σ^2

$\alpha=0.05$
b) ~~t_n~~ FOR μ;

$t_{n-1;\alpha/2}=t_{15;0.025}=2.13$

$t_{n-1;\alpha/2}\ S/\sqrt{n}=2.13\times 2/4=\frac{1}{2}\ 2.13=1.07$

CONFIDENCE INTERVAL FOR $\alpha=0.05$

$40-1.07<\mu<40+1.07$

12 OR $38.93<\mu<41.07$

FOR σ^2

$\chi^2_{n-1;\alpha/2}=\chi^2_{15;0.025}=27.49$

$\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}}=\frac{(15)(4)}{27.49}=2.18$ 2

$\chi^2_{n-1;1-\alpha/2}=\chi^2_{15;0.975}=6.26$

$\frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}=\frac{(15)(4)}{6.26}=9.60$

SO FOR $\alpha=0.05$, CONFIDENCE INTERVAL IS

$6.26<\sigma^2<9.60$

~~2.50<σ~~

$2.18<\sigma^2<9.60$

b) $H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

$\alpha = 0.01$

$\sigma_1^2 = \sigma_2^2$

FROM 9: (B) (7) -5

$\underline{t = 4.5}$

$t_{n_1+n_2-2;\alpha/2} = t_{16;.005} = 2.92$

is $t < -t_{\alpha/2}$? (is $4.5 < -2.92$) NO!

can't reject H_0 yet

is $t > t_{\alpha/2}$ (is $4.5 > 2.92$) YES!

Reject H_0

c) $\alpha = 0.02$

(B) $F = 9/4 = 2\frac{1}{4} = 2.25$

$F_{7;9;0.01} = 5.61$

is $F > F_{7;9:0.01}$?

is $2.25 > 5.61$? NO!

cannot reject H_0

4) $n=9$

$\sigma^2 = 4$

$H_0: \mu = 5$

$H_1: \mu < 5$

$\alpha = P[\text{TYPE 1 ERROR}]$

$= P[\text{Rejecting } H_0 \text{ given } \mu = 5]$

$= P[\bar{X} < 4 \text{ GIVEN } \mu = 5]$

$= P\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = Z < \frac{(4-5)3}{2}\right]$

$= P[Z < -1.5]$

$= 1 - P[Z < 1.5]$

$= 1 - 0.933$

$= .067$

$\beta = P[\text{TYPE II ERROR}]$

$= P[\text{accepting } H_0 \text{ given } \mu = 3.5]$

$= P[\bar{X} > 4]$

$= P\left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = Z > \frac{(4-\overset{0.5}{3.5})3}{2} \overset{1.5}{} = -0.75\right]$

$= P[Z < 0.75)$

$= 0.7734$

-2

Test 3 - MA 506 - Answers

1.

	freshman	sophomore	junior	senior	
Drinkers	30 $\boxed{40}$	30 $\boxed{25}$	20 $\boxed{20}$	20 $\boxed{15}$	100
Non-drinkers	50 $\boxed{40}$	20 $\boxed{25}$	20 $\boxed{20}$	10 $\boxed{15}$	100
	80	50	40	30	200

$$\chi^2 = \frac{(40-30)^2}{40} + \frac{(50-40)^2}{40} + \frac{(30-25)^2}{25} + \frac{(20-25)^2}{25}$$

$$+ \frac{(20-20)^2}{20} + \frac{(20-20)^2}{20} + \frac{(20-15)^2}{15} + \frac{(10-15)^2}{15}$$

$$= 10.33$$

$\chi^2_{3,.05} = 7.815$ Reject H_0.

2. $H_0: p = \frac{1}{3}$

$H_1: p < \frac{1}{3}$

$$z = \frac{318 - 333.333}{\sqrt{1000\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}} = \frac{-(15.333)(3)}{\sqrt{2000}} = \frac{-46}{44.7}$$

$$= -1.03$$

$-z_{.10} = -1.282$

Accept H_0

3.(a)

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
-2	-1	4	1	2
-1	-1	1	1	1
0	0	0	0	0
1	0	1	0	0
2	0	4	0	0
3	2	9	4	6
3	0	19	6	9

$\Sigma x_i = 3$

$\Sigma x_i^2 = 19$

$\Sigma y_i = 0$

$\Sigma y_i^2 = 6$

$\Sigma x_i y_i = 9$

3 (cont.) $\bar{x} = \frac{1}{2}$ $\bar{y} = 0$

$$\sum (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2 = 19 - 6 \cdot \tfrac{1}{4} = 17.5$$

$$\sum (y_i - \bar{y})^2 = \sum y_i^2 - n\bar{y}^2 = 6$$

$$\sum (x_i - \bar{x})(y_i - \bar{y}) = \sum x_i y_i - n\bar{x}\bar{y} = 9$$

$$b = \frac{9}{17.5} = .515 \qquad a = \bar{y} - b\bar{x} = -(.515)(\tfrac{1}{2}) = -.257$$

line is $y = -.257 + .515x$

$$SSR = b^2 \sum (x_i - \bar{x})^2 = (.515)^2 (17.5) = 4.65$$

$$SSD = SST - SSR = 6 - 4.65 = 1.35$$

$$S_e^2 = \frac{1.35}{4} = .34$$

$$S_b^2 = \frac{.34}{\sum (x_i - \bar{x})^2} = \frac{.34}{17.5} = .0195$$

$$S_b = .14$$

$$b \pm t_{4;.025} S_b = .515 \pm 2.776(.14) = .515 \pm .39 = \begin{cases} .125 \\ .905 \end{cases}$$

4.

B	B	B	A	B	B	B	A	A
20.6	21.6	21.9	22.1	22.2	22.5	23.1	23.3	23.7
1	2	3	4	5	6	7	8	9

B	A	B	A	A	A	B	A
24.0	24.1	24.2	24.8	25.4	26.1	26.2	26.[illegible]
10	11	12	13	14	15	16	17

$$R_1 = 4 + 8 + 9 + 11 + 13 + 14 + 15 + 17 = 91$$

$$U = (8)(9) + \frac{(8)(9)}{2} - 91 = 17 \qquad \mu_u = \frac{(8)(9)}{2} = 36$$

$$\sigma_u^2 = \frac{(8)(9)(18)}{12} = 108 \qquad \sigma_u = 10.4$$

$$z = \frac{17 - 36}{10.4} = -1.83 \qquad z_{.025} = 1.96 \quad \text{Accept } H_0$$

BOB MARKS

80

1) H_0:

40 | 30	25 | 30	20 | 20	15 | 20	100
40 | 50	25 | 20	20 | 20	15 | 10	100
80	50	40	30	200

$$\begin{array}{r} 2.67 \\ 3\overline{)8.00} \end{array} \qquad \begin{array}{r} 1.67 \\ 3\overline{)5.00} \end{array}$$

$\cancel{\frac{80 \times 100}{30} = \frac{800}{3}}$

$\cancel{\frac{50 \times 100}{30} = \frac{500}{3} = 167}$

$\cancel{\frac{40 \times 100}{20} = 200}$

$\cancel{\frac{30 \times 100}{20} = 150}$

100 8

$$\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{4} \frac{(f_{ij} - e_{ij})^2}{e_{ij}}$$

$$= \frac{(40-30)^2}{30} + \frac{(25-30)^2}{30} + 0 + \frac{(15-20)^2}{20}$$

not e_{ij}

$$+ \frac{(10)^2}{50} + \frac{(5)^2}{20} + 0 + \frac{(5)^2}{10}$$

$$= \frac{100}{30} + \frac{25}{30} + \frac{25}{20} + \frac{100}{50} + \frac{25}{20} + \frac{25}{10}$$

$$= 3.333 + 0.834 + 1.25 + 2 + 1.25 + 2.5$$

$$= 11.167$$

$$\chi^2_{3;0.005} = 12.838$$

CANNOT REJECT H_0

-7

3) $\sum X_i Y_i = 2 + 1 + 0 + 0 + 0 + 6 = 9$

$\sum X_i = 3 \Rightarrow \overline{X} = 1/2$

$\sum Y_i = 0 \Rightarrow \overline{Y} = 0$

$\sum X_i^2 = 4 + 1 + 1 + 4 + 9$
$= 19$

$\sum Y_i^2 = 1 + 1 + 4 = 6$

$$b = \frac{\sum X_i Y_i - n\overline{X}\,\overline{Y}}{\sum X_i^2 - n\overline{X}^2}$$

$$= \frac{9 - 6(\frac{1}{2})(0)}{19 - 6(1/4)} = \frac{9}{19 - 3/2} = \frac{9}{18\ 1/3} = \frac{9}{18.333}$$

17.5

$= 0.492$

$a = \overline{Y} - b\overline{X}$
$= 0 - (0.492)(\frac{1}{2})$
$= -0.246$

$\Rightarrow Y = -0.246 + 0.492X$

-2

b) $SSR = b^2 \sum (X_i - \overline{X}^2)$
$= b^2 [\sum X_i^2 - n\overline{X}^2]$
$= b^2 [19 - 6(\frac{1}{4})]$
$= 0.241 [19 - 3/2]$
$= (0.241)(18.333)$
$= 4.41$

$SST = \sum (Y_i - \overline{Y})^2 = \sum Y_i^2 - n\overline{Y}^2$
$= 6 + 0$

$SSD = \cancel{SST - SSR}$ $6.0 - 4.41$

-3

$= \cancel{4.41}$ 1.59

(4)

H_0: Identical Populations
H_1: Different Means

B	20.6	1
B	21.6	2
B	21.9	3
A	22.1	4
B	22.2	5
B	22.5	6
B	23.1	7
A	23.3	8
A	23.7	9
B	24.0	10
A	24.1	11
B	24.2	12
A	24.8	13
A	25.4	14
A	26.1	15
B	26.2	16
A	26.3	17

$n_1 = 8$
$n_2 = 9$

$\mu_U = \frac{n_1 n_2}{2} = \frac{8 \cdot 9}{2} = 4 \cdot 9 = 36$

$R_1 = \cancel{22.1}$, $\cancel{24.1}$, $\cancel{26.3}$, $\cancel{25.4}$, $\cancel{24.8}$, $\cancel{23.7}$, $\cancel{26.1}$, $\cancel{23.3}$, $\cancel{196.8}$

$R_1 = 4 + 8 + 9 + 11 + 13 + 14 + 15 + 17 = 92$

91

-5

$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - R_1$

$= 72 + \frac{8 \times 9}{2} - 92$

$= 72 + 28 - 92$ 100

$= 8 = U$

$\sigma_U^2 = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12} = \frac{9 \times 8 (18)}{12} = 108$

$\Rightarrow \sigma_\mu = 10.4$

$z = \frac{U - \mu_u}{\sigma_\mu} = \frac{8 - 36}{10.4} = \frac{-28}{10.4} = 2.69$

$\mp z_{\alpha/2} = \mp z_{0.025} = \mp 1.96$

$\Rightarrow$ REJECT H_0

THEM
8-2.3

INFERENCE

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INSERTS FOR
YOUR INDEX

CLASS SCHEDULE

TIME	1 7:50	2 8:45	3 9:40	4 10:35	5 11:30	6 12:25	7 1:20	8 2:15	9 3:10
MON.			SWITCH THEORY G308		STAT INF. G306				
TUE.			"		"				
WED.		ADV MATH II			"				
THU.			"						
FRI.			"		"			EL & MAG LAB	CO3
SAT.								EL & MAG LAB CO3	

FINAL EXAMINATIONS

DAY	DATE	TIME	PLACE	COURSE

NAME ______________________ TELEPHONE ____________

ADDRESS ______________________________________

SCHOOL ______________________ CLASS ____________

EFFICIENCY LINE ®

3-13-73

REVIEW

A) DISCRETE R.V.

1) BINOMIAL R.V. ⇒ THE PROB. THAT THERE WILL BE x SUCCESSES IN n TRIALS WITH EACH TRIAL HAVING PROBABILITY OF SUCCESS p

$$p_x(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \forall\, x = 0, 1, 2, \ldots, n$$

$$\mu_x = np \; ; \; \sigma_x^2 = np(1-p)$$

$p_x(x)$ IS THE PROBABILITY FUNCTION OR BINOMIAL DISTRIBUTION

2) BERNOULLI R.V. ⇒ A BINOMIAL R.V. FOR $n = 1$

$$P_x(x) = \begin{cases} p & , x = 1 \\ 1-p & , x = 0 \end{cases}$$

3) POISSON DIST.

$$P_x(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \forall\, x = 0, 1, 2, \ldots, n$$

$\lim_{n \to \infty}$ (BINOMIAL DIST) = POISSON DIST.

np = CONSTANT

LET P(ONE SUCCESS IN Δt) $= \alpha \Delta t$

THEN,

$$p(x,t) = \frac{e^{-\alpha t}(\alpha t)^x}{x!}$$

B) CONTINUOUS R. V.

1) NORMAL DISTRIBUTION

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \;\forall\, x \in (-\infty, \infty)$$

2) EXPONENTIAL R. V.

$$f_x(x) = \theta e^{-\theta x} \;\forall\, x \in [0, \infty)$$

C) EXPECTED VALUE

1) LET X BE DISCRETE WITH PROB FUNCTION $p_x(x)$

THEN

$$E[g(x)] = \sum_x g(x)\, p_x(x)$$

2) LET X BE CONTINUOUS WITH PDF $f_x(x)$ THEN:

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

3) $\mu = E[x]$

$$\sigma^2 = E[(x-\mu)^2]$$

$$= E(x^2) - \mu^2$$

D) SAMPLING DISTRIBUTION IS A DENSITY OR PROBABILITY FUNCTION OF A STATISTIC SUCH THAT

$\exists\, \bar{X}$ OR σ^2, ETC.

DEFINITION: LET k BE A POSITIVE INTEGER. THE k^{TH} MOMENT OF A R.V. ABOUT THE ORIGIN IS:

$$m_k = E[X^k]$$

THE k^{TH} MOMENT ABOUT THE MEAN IS:

$$E[(X-\mu)^k]$$

DEFINITION: THE MOMENT GENERATING FUNCTION FOR A R.V. X IS:

$$M_X(t) = E[e^{tX}]$$

PROVIDED THAT THE EXP. VALUE EXISTS FOR SOME OPEN INT. OF VALUE ABOUT $t = 0$

IF X IS FINITE:

$$M_X(t) = \sum_x e^{tx} p(x)$$

IF X IS CONTINUOUS

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

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MOMENT GENERATING FUNCTION

$$M_X(t) = E[e^{tx}] = \begin{cases} \sum_x e^{tx} p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \end{cases}$$

THEOREM: UNDER CERTAIN REGULARITY CONDITIONS

$$m_k = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}$$

PROOF: LET X BE CONTINUOUS WITH P.D.F. $f(x)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$M_X'(t) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \frac{d}{dt}(e^{tx})\,dx$$

$$= \int_{-\infty}^{\infty} x e^{tx} f(x)\,dx$$

$$M_X''(t) = \int_{-\infty}^{\infty} x^2 e^{tx} f(x)\,dx$$

$$M_X^{(k)}(t) = \int_{-\infty}^{\infty} x^k e^{tx} f(x)\,dx$$

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = M_X^{(k)}(0) = \int_{-\infty}^{\infty} x^k f(x)\,dx = E[X^k]$$

$$= m_k$$

$\ni$ m_k IS THE K^{TH} MOMENT ABOUT THE ORIGIN

Q.E.D.

MACLAURIN SERIES EXPANSION

$$f(t) = f(0) + f'(0)\frac{t}{1!} + \frac{f''(0)t^2}{2!} + \frac{f'''(0)t^3}{3!} + \dots$$

$$\therefore M_X(t) = \sum_{n=0}^{\infty} \frac{M_X^{(n)}(0)\,t^n}{n!}$$

$$M_X(0) = 1$$

$$\Rightarrow M_X(t) = 1 + \frac{m_1 t}{1!} + \frac{m_2 t^2}{2!} + \dots$$

$$M_X(t) = 1 + \sum_{n=0}^{\infty} \frac{m_n t^n}{n!}$$

MOMENTS FOR BINOMIAL DISTRIBUTION

$$b(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$M_x(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

NOW $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$

$$\Rightarrow M_x(t) = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= [pe^t + 1 - p]^n$$

$$= [p(e^t - 1) + 1]^n$$

FINDING MOMENTS:

$$M_x'(t) = n(pe^t + 1 - p)^{n-1} pe^t$$

$$= np[(pe^t + 1 - p)^{n-1} e^t]$$

$$M_x'(t) = np[(n-1)e^t (pe^t + 1 - p)^{n-2} pe^t + (pe^t + 1 - p)^{n-1} e^t]$$

$$M_x'(0) = m_1 = np$$

$$M_x''(0) = np[(n-1)p + 1]$$

$$= n(n-1)p^2 + np = m_2 = E[x^2]$$

RECALL $\sigma^2 = E[x^2] - \mu^2$

$$\mu^2 = E[x] = m_1$$

FOR BINOMIAL THEN;

$$\sigma^2 = np(1-p)$$

STANDARDIZED NORMAL DISTRIBUTION

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx)} dx$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2tx + t^2)} e^{\frac{1}{2}t^2} dx$

$= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} dt$

$= e^{\frac{1}{2}t^2}$

RECALL $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$

$= \sum_n \frac{x^n}{n!}$

ERGO $e^{\frac{1}{2}t^2} = 1 + (\frac{1}{2}t)^2 + \frac{(\frac{1}{2}t^2)^2}{2!} + \ldots$

$= \sum_k \frac{(\frac{1}{2}t^2)^k}{k!}$

$= 1 + (0)t + \frac{1}{2}t^2 + 0 \times t^3$

$+ (\frac{1}{2})^2 \frac{t^4}{2!} + 0 \cdot t^5 + (\frac{1}{2})^3 \frac{t^6}{3!} + \ldots$

$\Rightarrow m_k = 0$ FOR ALL ODD k

$\therefore e^{\frac{1}{2}t^2} = 1 + (\frac{1}{2}) \cdot \frac{2!\, t^2}{2!} + (\frac{1}{2})^2 \frac{t^4}{k} \frac{4!}{2!}$

$+ (\frac{1}{2})^3 \frac{t^6}{6!} \frac{6!}{3!} + \ldots$

$\Rightarrow m_0 = 1$

$m_2 = \frac{1}{2}\, 2!\, \frac{t^2}{2!}$

$m_4 = (\frac{1}{2})^2 \frac{4!}{2!} \frac{t^4}{n!}$

$m_6 = (\frac{1}{2})^3 \frac{6!}{3!} \frac{t^6}{6!}$

$\Rightarrow m_{2n} = (\frac{1}{2})^n \frac{(2n)!}{n!}$

MOMENT GENERATING FUNCTIONS DON'T ALWAYS EXIST

EX) $f(x) = \begin{cases} 1/x^2 & \text{FOR } x \geq 1 \\ 0 & \text{OTHERWISE} \end{cases}$

$$m_1 = \mu = E[X] = \int_1^\infty x \frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \infty$$

NOW $M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{x^2}\,dx$

3-16-73

CONT) $f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & \text{OTHERWISE} \end{cases}$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_1^\infty e^{tx} \frac{1}{x^2}\,dx$$

$$= \int_1^\infty \left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots\right] \frac{1}{x^2}\,dx$$

$$= \int_1^\infty \left[\frac{1}{x^2} + \frac{t}{x} + \frac{t^2}{2!} + \frac{t^3 x}{3!} + \dots\right] dx$$

LET $t > 0$

$$\Rightarrow M_x(t) > \int_1^\infty \left[\frac{1}{x^2} + \frac{t}{x}\right] dx = \infty$$

$\therefore M_x(t)$ INTEGRAL DIVERGES

THEOREM: LET $Y = aX + b$, WHERE a AND b ARE CONSTANT AND $M_X(t)$ EXISTS, THEN

$$M_Y(t) = e^{bt} M_X(at)$$

PROOF: $M_Y(t) = E[e^{Yt}]$

$$= E[e^{(aX+b)t}]$$

$$= E[e^{bt} e^{atX}]$$

$$= e^{bt} E[e^{Xat}]$$

$$= e^{bt} M_X(at)$$

RECALL $M_Z(t)$ OF $N(0,1) = e^{\frac{1}{2}t^2}$

AND $Z = \frac{X-\mu}{\sigma}$

$\Rightarrow X = \sigma Z + \mu$

SO THAT $M_X(t)$ OF $N(\mu, \sigma^2)$ IS

$$M_X(t) = e^{\mu t} M_Z(\sigma t)$$

FROM THE PREVIOUS THEOREM. THUS.

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

COROLLARY 1: LET X HAVE MEAN μ AND M.G.F. OF $M_X(t)$.

THEN: $M_{X-\mu}(t) = e^{-\mu t} M_X(t)$

COROLLARY 2: LET X HAVE MEAN μ, STANDARD DEVIATION σ, AND M.G.F. $M_X(t)$. THEN:

$$M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu t}{\sigma}} M_X(t/\sigma)$$

PROBLEMS Pp. 97-98: 1, 2, 3

Pg 112: 10

Pg 129: 7 (ALSO STANDARD FORM)

Pg 136: 3

FOR POISSON: $P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

JOINT PROBABILITY FUNCTION

$p(x, y) = P[X = x \text{ AND } Y = y]$

$$P_{\underline{X}}(\underline{x}) = P_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$$
$$= P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]$$

TWO R.V.

$f(x, y) \Rightarrow$ DENSITY FUNCTION

$$P[(X, Y) \in R] = \iint_R f(x, y)\, dx\, dy$$

n R.V.

$$f_{\underline{X}}(\underline{x}) = f_{X_1, X_2, X_3, \dots, X_n}(x_1, x_2, \dots, x_n)$$

$$f_{X_1}(x_1) = \underbrace{\int\int\int_{-\infty}^{\infty} \cdots \int}_{n-1} f_{\underline{X}}(\underline{x})\, dx_2\, dx_3 \dots dx_n$$

DEFINITION: $X_1, X_2, \dots, X_n$ ARE INDEPENDENT DISCRETE RANDOM VARIABLES IFF

$$P_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} P_{X_i}(x_i)$$

AND $X_1, X_2, \dots, X_2$ ARE INDEPENDENT CONTINUOUS RANDOM VARIABLES IFF

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

DEFINITION: A SET OF RANDOM VARIABLES ARE IDENTICALLY DISTRIBUTED IFF ALL OF THEIR DENSITY OR PROBABILITY FUNCTIONS ARE THE SAME.

THEOREM: IF $X_1, X_2, \ldots, X_n$ ARE INDEPENDENT WHOSE M.G.F. EXIST AND $Y = \sum_{i=1}^{n} a_i X_i$ WHERE $a_1, a_2, \ldots$ ARE CONSTANTS

THEN

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

[illegible]

COROLLARY: IF $X_1, X_2, \ldots, X_n$ ARE IDENTICALLY DISTRIBUTED RANDOM VARIABLES HAVE COMMON M.G.F. $M_X(t)$ AND IF $Y = \sum_{i=1}^{n} X_i$ THEN $M_Y(t) = [M_X(t)]^n$

THEOREM: IF TWO RANDOM VARIABLES HAVE THE SAME M.G.F., THEY HAVE THE SAME CUMULATIVE DISTRIBUTION FUNCTIONS

3-19-73

Pg 112, #10

$$M_{\frac{x-\mu}{\sigma}}(t) = e^{-\frac{\mu}{\sigma}t} M_x\left(\frac{t}{\sigma}\right)$$

$$M_x(t) = (pe^t + q)^n \quad \ni q = 1 - p$$

$$\mu = np \quad ; \sqrt{npq}$$

$$\Rightarrow M_{\frac{x-\mu}{\sigma}}(t) = e^{-\frac{\mu}{\sigma}t}\left(pe^{\frac{t}{\sigma}} + q\right)^n$$

$$= e^{-\frac{np}{\sqrt{npq}}t}\left(e^{\frac{t}{\sqrt{npq}}} + q\right)^n$$

$$= e^{-\sqrt{\frac{np}{q}}t}\left(pe^{\frac{t}{\sqrt{npq}}} + q\right.$$

P_8 129

c) $f(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, 3, \ldots$

$$M_X(t) = E[e^{Xt}]$$

$$= \sum_{x=0}^{\infty} e^{Xt} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda}(\lambda e^t)^x / x!$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

BERNOULLI R.V.; $X_1, X_2, \ldots, X_n$ $\quad X_i = 1$ OR 0

Y = SUCCESSES IN Y TRIALS

$\therefore Y = \sum_{i=1}^{n} X_i$

THEOREM: IF $X_1, X_2, \ldots, X_n$ ARE INDEPENDENT BERNOULLI R.V. WITH PARAMETER P, THEN $Y = \sum_{i=1}^{n} X_i$ IS A BINOMIAL R.V. WITH PARAMETER'S.

PROOF: EACH X_i HAS A M.G.F. = $(pe^t + q)$

THUS $M_Y(t) = (pe^t + q)^n$

THEOREM: IF $X_1, X_2, \ldots X_n$ ARE INDEPENDENT NORMAL RANDOM VARIABLES WITH MEANS μ_i AND VARIANCES σ_i^2 AND IF $a_1, a_2, \ldots, a_n$ ARE CONSTANTS, ~~THEN $Y = \sum_{i=1}^{n} X_i$~~ THEN $Y = \sum_{i=1}^{n} a_i X_i$ IS A NRV WITH MEAN $\mu_Y = \sum_{i=1}^{n} a_i \mu_i$ AND VARIANCE $\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

PROOF: $M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(a_i t)$$

$$= \prod_{i=1}^{n} e^{\mu_i a_i t + \frac{1}{2}\sigma_i^2 a_i^2 t^2}$$

$$= e^{t[\Sigma a_i \mu_i] + \frac{1}{2}t^2[\Sigma a_i^2 \sigma_i^2]}$$

$$= e^{\mu_Y t + \frac{1}{2}t^2 \sigma_Y^2}$$

(WORKS ALSO FOR ALL INDEPENDENT RANDOM VARIABLES)

PROOF: $Y = \sum_{i=1}^{n} a_i X_i$

$$\mu_Y = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu_i$$

$$\sigma_Y^2 = E[(Y - \mu_Y)]^2$$

$$= E\left\{\left[\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \mu_i\right]^2\right\}$$

$$= E\left\{\left[\sum_{i=1}^{n} a_i (X_i - \mu_i)\right]^2\right\}$$

RECALL

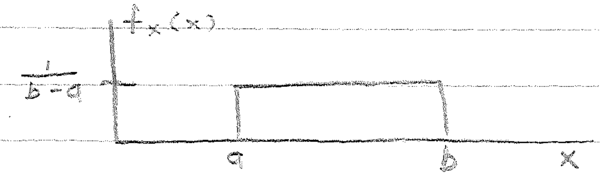
$$\left(\sum_{i=1}^{n} b_i\right)^2 = \sum_{i=1}^{n} b_i^2 + 2\sum_{i<j}\sum b_i b_j$$

$$\Rightarrow \sigma_Y^2 = E\left\{\sum_{i=1}^{n} a_i^2 (X_i - \mu_i)^2 + 2\sum_{i<j}\sum (X_i - \mu_i)(X_j - \mu_j)\right\}$$

$$= \sum_{i=1}^{n} a_i^2$$

PROBLEMS: Pg 179-80: 1, 3, 4 (PROVE & GENERALIZE) 7, 8, 13

3-2 - 72



$$M_x(t) = \frac{1}{t(b-a)}\left[e^{bt} - e^{at}\right]$$

$$M_x'(t) = \frac{1}{b-a}\,\frac{t(be^{bt} - ae^{at}) - (e^{bt} - e^{at})}{t^2}$$

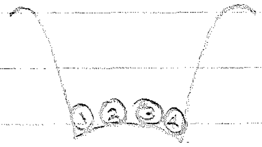
$$M_x(t) = \frac{1}{t(b-a)}\left[\frac{\left(1 + bt + \frac{(bt)^2}{2!} + \dots\right) - \left(1 + at + \frac{(at)^2}{2!} + \dots\right)}{1}\right]$$

$$= \frac{1}{b-a}\left[(b-a) + \frac{(b^2-a^2)t}{2!} + \frac{(b^3-a^3)t^2}{3!} + \frac{(b^4-a^4)t^3}{4!} + \dots\right]$$

$$= 1 + \left(\frac{a+b}{2}\right)t + \frac{b^2+ab+b^2}{3}\frac{t^2}{2!} + \frac{b^3+a^2b+ab^2+a^3}{4}\frac{t^3}{3!} + \dots$$

$$\therefore \quad m_1 = \frac{a+b}{2}$$

$$m_2 = \frac{b^2+ab+b^2}{3}$$



LET X BE THE NUMBER WE GET IF WE PICK ONE NUMBER FROM THE BOWL

x	$p_x(x)$
1	1/4
2	1/4
3	1/4
4	1/4

$$\mu_x = E[x] = \sum_x x\,p(x) = 5/2$$

$$\sigma_x^2 = E[x^2] - \mu_x^2$$

$$E[x^2] = \sum_x x^2 p(x) = \left[(1)^2 + (2)^2 + (3)^2 + (4)^2\right]\tfrac{1}{4} = 15/2$$

$$\sigma_x^2 = \frac{30}{4} - \left(\frac{5}{2}\right)^2 = 5/4$$

SUPPOSE WE TAKE A RANDOM SAMPLE OF 2 WITH REPLACEMENT FROM THE BOWL. LET X_1 BE THE FIRST NUMBER CHOSEN AND X_2 THE SECOND. $p_1(x_1) = 1/4$; $p_2(x_2) = 1/4$

(x_1, x_2)	$p(x_1, x_2)$
1 1	1/16
1 2	1/16
1 3	1/16
1 4	1/16
2 1	1/16
2 2	1/16
2 3	1/16
2 4	1/16
3 1	1/16
3 2	1/16
3 3	1/16
3 4	1/16
4 1	1/16
4 2	1/16
4 3	1/16
4 4	1/16

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DEFINITION: THE SET OF RANDOM VARIABLES $X_1, X_2, \ldots, X_n$ ARE CALLED A RANDOM SAMPLE OF THE RANDOM VARIABLE X IFF THEY ARE INDEPENDENT AND IDENTICALLY DISTRIBUTED AS X

2-22-73

x_1	x_2	x_1	x_2
1	1	3	1
1	2	3	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
2	2	4	2
2	3	4	3
[illegible]	[illegible]	4	4

$\mu = $ [illegible]

$\sigma^2 = 3/4$

DEFINITION: A STATISTIC IS A FUNCTION OF A RANDOM SAMPLE OF A RANDOM VARIABLE X WHICH DOES NOT DEPEND ON ANY UNKNOWN PARAMETERS OF X. (THE SAME FUNCTIONS OF THE OUTCOMES OF THE RANDOM SAMPLE IS ALSO CALLED A STATISTIC)

DEFINITION: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF THE RANDOM VARIABLE X. THEN THE SAMPLE MEAN RANDOM VARIABLE IS $\bar{X} = \sum X_i / n$. THE OUTCOME OF $\bar{X}$, NAMELY $\bar{x}$, IS THE SAMPLED MEAN

$x_1 x_2$	$\bar{x}$	$x_1 x_2$	$\bar{x}$
11	1	31	2
12	3/2	32	5/2
13	2	33	3
14	5/2	34	7/2
21	3/2	41	5/2
22	2	42	3
23	5/2	43	7/2
24	3	44	4

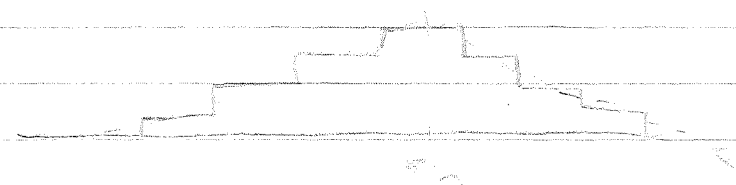
$\bar{x}$	$p_{\bar{x}}(\bar{x})$
1	1/16
3/2	2/16
2	3/16
5/2	4/16
3	3/16
7/2	2/16
4	1/16

$\mu_{\bar{x}} = E[\bar{X}] = \sum \bar{x}\, p_{\bar{x}}$

$= 5/2$

$E[\bar{X}^2] = \sum \bar{x}^2 p_{\bar{x}}(\bar{x}) = \frac{5}{?}$

$\sigma_{\bar{x}}^2 = \frac{\sigma_x^2}{n}$



THEOREM: LET $X_1, X_2, \ldots, X_n$ BE [illegible]

[illegible]

X WITH [illegible] A [illegible]

[illegible] $\bar{X}_n = \sum_{i=1}^{n} X_i / n$,

[illegible] $\mu_{\bar{X}_n} = \mu_X$ AND $\sigma^2_{\bar{X}} = \sigma^2_X / n$.

[illegible]

[illegible] $\bar{X}_n$ IS A NORMAL

RANDOM VARIABLE.

PROOF: $\bar{X}_n = \sum_{i=1}^{n} X_i / n$

$= \sum_{i=1}^{n} \frac{X_i}{n}$

[illegible]

$= \sum_{i=1}^{n} \frac{1}{n} \mu = \mu$

AND

$\sigma^2_{\bar{X}} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2_X$

$= \frac{n}{n^2} \sigma^2_X = \sigma^2_X / n$

BY THE THEOREM [illegible] THAT

[illegible] NORMAL [illegible]

THEOREM (CHEBYSHEV'S INEQUALITY) IF X IS A RANDOM VARIABLE WITH MEAN μ_x AND VARIANCE σ_x^2 AND $k > 0$, THEN

$$P\left[|X-\mu_x| \geq k\sigma\right] \leq 1/k^2$$

(OR IF $c > 0$, $P\left[|X-\mu_x| \geq c\right] \leq \sigma_x^2/c^2$)

PROOF: FOR X [illegible]

[illegible]

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$\geq \int_{\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx + \int_{\mu+k\sigma}^{k^2\sigma^2} f(x)\,dx$$

$$= k^2\sigma^2\left[P(X < \mu-k\sigma) + P\left[X > \mu+k\sigma\right]\right]$$

$$= k^2\sigma^2 P\left[X < \mu-k\sigma \text{ OR } X > \mu+k\sigma\right]$$

$$= k^2\sigma^2 P\left[|X-\mu| > k\sigma\right]$$

$$\Rightarrow P\left[|X-\mu| \geq k\sigma\right] \leq 1/k^2$$

3-23-73

CHEBYCHEV'S THEM:

X HAS MEAN μ_x AND VARIANCE σ_x^2 AND $k>0$

$$\Rightarrow P[|X-\mu_x| \geq k\sigma_x] \leq 1/k^2$$

FOR ANY $c>0$, $P[|X-\mu_x| \geq c] \leq \frac{\sigma_x^2}{c^2}$

EXAMPLE

LET $X \sim N(\mu, \sigma^2)$

$k=1$:

$$P[|X-\mu| \geq \sigma] \leq 1$$

$k=2$

$$P[|X-\mu| \geq 2\sigma] \leq 1/4$$

$k=3$

$$P[|X-\mu| \geq 3\sigma] \leq 1/9$$

$k=1$: $P[|X-\mu| \geq \sigma] = P\left[\frac{|X-\mu|}{\sigma} \geq 1\right]$

$= P[|Z| \geq 1]$

$= 0.3179 \leq 1$

$k=2$: $P[|Z| \geq 2] = 0.0454 \leq 1/4$

$k=3$: $P[|Z| \geq 3] = 0.0026 \leq 1/9$

COROLLARY 1: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF THE RANDOM VARIABLE X, WHICH HAS MEAN μ AND VARIANCE σ_x^2. LET $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. THEN FOR ANY $\epsilon > 0$:

$$\lim_{n\to\infty} P[|\bar{X}_n - \mu| \geq \epsilon] = 0$$

20

PROOF: FOR ANY $\epsilon > 0$, $P[|\bar{X}_n - \mu_{\bar{X}_n}| \geq \epsilon] \leq \frac{\sigma^2_{\bar{X}_n}}{\epsilon^2}$

FOR ANY $\epsilon > 0$; $P[|\bar{X}_n - \mu| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2} \; \forall n$

NOW IF: $0 \leq a_n \leq b_n$

THEN $0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

$\Rightarrow 0 \leq P[|\bar{X}_n - \mu| \geq \epsilon] = \frac{\sigma^2}{n\epsilon^2}$

$\therefore 0 \leq \lim_{n\to\infty} P[|\bar{X}_n - \mu| \geq \epsilon] \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$

$\Rightarrow \lim_{n\to\infty} P[|\bar{X}_n - \mu| \geq \epsilon] = 0$

COROLLARY 2: LET Y_n BE THE NUMBER OF SUCCESSES IN A BINOMIAL EXPERIMENT WITH PARAMETERS n & p. THE $\forall \; \epsilon > 0$,

$\lim_{n\to\infty} P[|\frac{Y_n}{n} - p| \geq \epsilon] = 0$

THEOREM M: LET $X_1, X_2, \ldots X_n$ BE A SEQUENCE OF R.V. WITH CDF OF $F_{X_n}(t)$, MGF $= M_{X_n}(t)$, IF THERE EXIST A RV X WITH CDF $F_X(t)$ AND $M_X(t)$ SUCH

$\lim_{n\to\infty} M_{X_n}(t) = M_X(t)$. THEN

$\lim_{n\to\infty} F_{X_n}(t) = F_X(t)$.

THEOREM ~~6.6.1~~: CENTRAL LIMIT THEM.

LET $X_1, X_2, \ldots, X_n$ BE A SEQUENCE OF IDENTICALLY DISTRIBUTED INDEPENDENT R.V.'S, EACH WITH FINITE MEAN μ AND FINITE VARIANCE σ^2, LET $S_n = \sum_{i=1}^{n} X_i$, $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i = S_n/n$, AND LET $Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$,

LET $N_Z(t)$ BE THE C.D.F. OF THE STANDARDIZED NORMAL R.V. AND $F_{Z_n}(t)$ BE THE C.D.F. THEN FOR ALL t:

$$\lim_{n\to\infty} F_{Z_n}(t) = N_Z(t)$$

3-25-73

CENTRAL LIMIT THEM. DICTATES

$$S_n = \sum_{i=1}^{n} X_i$$

$$Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \longrightarrow Z$$

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \longrightarrow Z$$

COROLLARY: LET Y_n BE A BINOMIAL R.V. WITH PARAMETERS n AND p. THEN IF $Z_n = Y_n/n - p \Big/ \sqrt{\frac{p(1-p)}{n}} = \frac{Y_n - np}{\sqrt{np(1-p)}}$ THEN

$$\lim_{n\to\infty} F_{Z_n}(t) = N_Z(t)$$

SUCH THAT $N_Z(t)$ IS THE C.D.F OF THE $N(0,1)$ R.V.

PROOF: $Y_n = \sum_{i=1}^{n} X_i$, WHERE X_i ARE IDENTICAL BERNOULLI R.V.

TO BE PROVED:

THEOREM: LET $P_{X_\lambda}(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ FOR $x = 0, 1, 2, \ldots$ AND LET $Z_\lambda = (X_\lambda - \lambda)/\sqrt{\lambda}$.

LET $F_\lambda(t)$ BE THE C.D.F. OF Z_λ. THEN (FROM THE M.G.F. OF Z_λ),

$$\lim_{\lambda\to\infty} F_\lambda(t) = N_Z(t)$$

HINTS:

CHECK $\lim_{n\to\infty} \ln M_{Z_n}(t) = \frac{1}{2}t^2 = \ln \lim_{n\to\infty} M_{Z_n}(t)$

CHECK CENTRAL LIMIT THEM. PROOF

TOSS A COIN 20 TIMES, LET X BE THE NUMBER OF HEADS IN 20 TOSSES

FIND $P[8 \le X \le 11]$; $p = 1/2$; $n = 20$

(A) FROM BINOMIAL R.V. TABLES (P. 355)

$$P[8 \le X \le 11] = P[X \le 11] - P[X \le 7]$$
$$= 0.7483 - 0.1316$$
$$= 0.6167$$

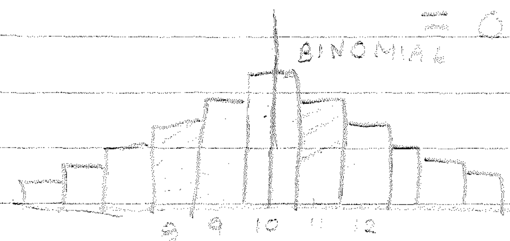
(B) FROM NORMAL R.V. TABLES

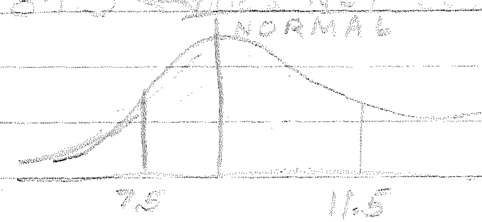
$\mu = np = 10$

$\sigma_x^2 = np(1-p) = 5$

$\sigma_x = \sqrt{5} \approx 2.25 = 9/4$

$$P[8 \le X_B \le 11] = P[8 \le X_N \le 11]$$
$$= P\left[\frac{8-10}{9/4} \le Z \le \frac{11-10}{9/4}\right]$$
$$= P[-0.89 \le Z \le 0.44]$$
$$= 0.6700 - 0.1867$$
$$= 0.4843 \Leftarrow \text{DOES NOT COMPARE}$$





NEED $\frac{1}{2}$ CORRECTION FACTOR

$$P[8 \le X_B \le 11] = P[7.5 \le X_N \le 11.5]$$
$$= P\left[\frac{7.5-10}{9/4} \le Z_N \le \frac{11.5-10}{9/4}\right]$$
$$= P\left[-10/9 \le Z \le 6/9\right]$$
$$= P[-1.11 \le Z \le 0.67]$$
$$= 0.7486 - 0.1335$$
$$= 0.6151 \Leftarrow \text{MUCH BETTER APPROX.}$$

X IS POISSON WITH $\lambda = 15$; $X_P \sim X_N \sim N(15, \overset{\sigma}{\sqrt{15}})$

FIND $P[11 \le X_P \le 19] \sim P[10.5 \le X_P \le 19.5]$

$$= P\left[\frac{10.5-15}{3.9} < Z < \frac{19.5-15}{3.9}\right]$$

$$= P[-1.15 \le Z \le 1.15]$$

$$= 2(0.3749)$$

$$\approx 0.7498$$

USING POISSON R.V.

$$P[11 \le X_P \le 19] = 0.875 - 0.118$$

$$= 0.757$$

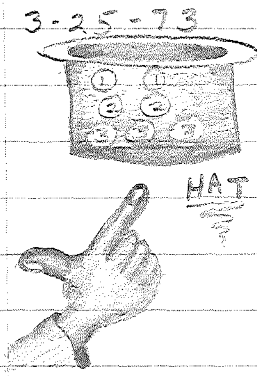
PRETTY CLOSE

HOMEWORK: (PROOF) pp. 193-4 ; 1-10 (ALL)

pp. 185-6 ; 1, 2, 5, 6

TEST ON FRIDAY (CLOSED BOOK)

3-25-73



x	$p(x)$
1	$2/7$
2	$2/7$
3	$3/7$

A STATISTIC! $\longrightarrow \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

DEFINITION: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF THE RANDOM VARIABLE X.

a) THE SAMPLE MEAN RANDOM VARIABLE IS

$$\bar{X} = \sum_{i=1}^{n} X_i \Big/ n$$

b) THE SAMPLE VARIANCE RANDOM VARIABLE IS:

$$S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$

c) THE SAMPLE STANDARD DEVIATION R.V. IS

$$S = \sqrt{S^2}$$

EXAMPLE: DATA $\Rightarrow 3, -1, 2, 4$

$$\bar{x} = \frac{1}{4}(3 - 1 + 2 + 4) = 2$$

OUTCOME OF A RANDOM VARIABLE $\rightarrow s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} = \frac{1}{3}\left[(3-2)^2 + (-1-2)^2 + (2-2)^2 + (4-2)^2\right]$

$$= 14/3$$

$$s = \sqrt{14/3}$$

DEFINITION: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF X AND LET $x_1, x_2, \ldots, x_n$ BE THE OUTCOME OF $X_1, X_2, \ldots, X_n$.

RANK THE OUTCOMES $x_1, x_2, \ldots, x_n$ IN ORDER OF INCREASING MAGNITUDE, YIELDING $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ WHERE $x_{(1)}$ IS THE SMALLEST ELEMENT AND $x_{(n)}$ IS THE LARGEST

a) THE RANDOM VARIABLE $X_{(i)}$ WHOSE OUTCOME IS $x_{(i)}$ IS CALLED THE i^{TH} ORDER STATISTIC R.V.

b) $X_{(1)}$ IS THE SMALLEST SAMPLE VALUE R.V.

c) $X_{(n)}$ IS THE LARGEST SAMPLE VALUE R.V.

d) $$M_0 = \begin{cases} X_{\left(\frac{n+1}{2}\right)} & \text{FOR ODD } n \\ \frac{1}{2}\left[X_{\left(\frac{n}{2}\right)} + X_{\left(\frac{n}{2}+1\right)}\right] & \text{FOR EVEN } n \end{cases}$$

IS THE SAMPLE MEDIAN R.V.

EX) $x_i = 3, -1, 2, 4$

$$x_{(1)} = -1 \,;\; x_{(2)} = 2 \,;\; x_{(3)} = 3 \,;\; x_{(4)} = 4$$

$$m_0 = \frac{x_{(2)} + x_{(3)}}{2} = 5/2$$

e) $R = X_{(n)} - X_{(1)}$ IS THE SAMPLE RANGE RANDOM VARIABLE

EX) $x_i = 3, -1, 2, 4$

$$r = x_{(4)} - x_{(1)} = 4 - (-1) = 5$$

4

EXAM #1 MATERIAL

KNOW PROPERTIES OF m.g.f. (M_x)

MORE THEORY

ASSGN. 202-3; 1-9

209-10; 1-5, (NO SAMPLE DIST. FUNCTION)

3-28-73

Pg 189, #6

$P(x) = p(1-p)^{x-1}$ FOR $x = 1, 2, 3, \ldots$

$\mu = E[X] = \frac{1}{p}$

Pg 193 #3

$f_x = \frac{1}{5}$ FOR $15 \le x \le 20$

$E[x] = 17.5 \Rightarrow \mu_{\bar{x}} = 17.5$

$\sigma^2 = 25/12$

$\sigma = 5/2\sqrt{3} = \frac{5}{6}\sqrt{3} = 1.44 \Rightarrow \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.44}{\sqrt{250}} = 0.0091$

FIND $P[18 \le \bar{X} \le 20]$ ← NORMAL R.V.

$= P\left[\frac{18-17.5}{0.0091} \le z \le \frac{20-17.5}{0.0091}\right]$

Pg 199 #5

$f(x) = \lambda e^{-\lambda x}$ FOR $x > 0$

$\mu = E(x) = 1/\lambda = 1000 \Rightarrow \mu_{\bar{x}} = 1000.00$

$\sigma_x^2 = 1/\lambda^2 = (1000)^2 \Rightarrow \sigma_x = 1000 \Rightarrow \sigma_{\bar{x}} = \frac{\sigma_x}{n} = \frac{1000}{\sqrt{100}} = 100$

$P[\bar{x} > 950] = P\left[Z > \frac{950-1000}{100}\right]$

$= P[Z > 0.5]$

Pg 203 #9

$P(x) = p(1-p)^{x-1}$ $x = 1, 2, 3, \ldots$

$P(X_1, X_2, \ldots, X_{10}) = \prod_{i=1}^{10} p(1-p)^{x_i - 1}$; $x_i \ge 1 \; \forall i$

$= p^{10}(1-p)^{\sum_{i=1}^{10}(x_i - 10)}$

Pg 180, #6

$n_1 p; n_2 p; n_3 p, \ldots, n_k p$

$M_{x_i}(t) = (pe^t + q)^{n_i}$

$M_Y(t) = M_{\sum_{i=1}^{k} x_i}(t) = \prod_{i=1}^{k} (pe^t + q)^{n_i}$

$= (pe^t + q)^{\sum_{i=1}^{k} n_i}$

EXAM: MOMENT GENERATING FUNCTIONS, CHEBSHEV INEQUALITY, LAWS OF LARGE NUMBERS,

CENTRAL LIMIT THEM, ;

1-2-73

TEST ANSWERS

1) DATA $-1, -2, 0, 1, -3$

X_i	X_i^2
-1	1
-2	4
0	0
1	1
-3	9
-5	15

$\bar{X} = -5/5 = -1 \Leftarrow$ MEAN

$s^2 = \overline{X^2} - n\bar{X}^2$

$= \dfrac{15 - 5(-1)^2}{4}$

$= 5/2 \Leftarrow$ VARIANCE

$s = \sqrt{5/2} \Leftarrow$ STANDARD DEVIATION

$-3, -2, -1, 0, 1$

$X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ ORDER STATISTICS

$X_{(3)} = -1 \Leftarrow$ MEDIAN

$(1 - (-3)) = 4 \Leftarrow$ RANGE

2) $\mu = 2000$; $\sigma = 2000$; $n = 100$

FIND $P[\bar{X} > 2100]$; $\sigma_{\bar{x}} = \sigma/\sqrt{n} = 200$

$$= P\left[Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > \frac{2100-2000}{200}\right]$$

$$= P[Z > 1/2]$$

3) INDEPENDENT $\Rightarrow$ JOINT P.D.F. $= \prod(\text{PDF})$

a) $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$$f(x_1, x_2, x_3) = f_1(x_1)\, f_2(x_2)\, f_3(x_3)$$

$$= \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_1} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 + \left(\frac{x_1-\mu_3}{\sigma_3}\right)^2\right]}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{2} \times \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{6}}$$

$$\times e^{-\frac{1}{2}\left(\frac{x_1-20}{2}\right)^2} \times e^{-\frac{1}{2}\left(\frac{x_2-10}{\sqrt{3}}\right)^2}$$

$$\times e^{-\frac{1}{2}\left(\frac{x_3-15}{\sqrt{6}}\right)^2}$$

b) $Y = \sum a_i X_i$

$X_i \sim N(\mu_i, \sigma_i^2)$; $Y \sim N(\mu_Y, \sigma_Y^2)$

$\mu_Y = \sum a_i \mu_i$; $\sigma_Y^2 = \sum a_i^2 \sigma_i^2$

$\mu_Y = (3)(20) - (2)(10) + 5(15) = 115$

$\sigma_Y^2 = 3^2(4) + (-2)^2 3 + (5)^2(6) = 198$

4) a) IN BOOKS OR NOTES

b) $M_x(t) = e^{\lambda(e^t - 1)}$

$M_x'(t) = \lambda e^t e^{\lambda(e^t - 1)}$

$M_x'(0) = \lambda$

c) $Z_\lambda = (X - \lambda)/\sqrt{\lambda}$

$$M_Z(t) = e^{-\frac{\mu}{\sigma}t} M_x\left(\frac{t}{\sigma}\right)$$

$$= e^{-\sqrt{\lambda}t} M_x\left(\frac{t}{\sqrt{\lambda}}\right)$$

$$= e^{-\sqrt{\lambda}t} e^{\lambda(e^{t/\sqrt{\lambda}} - 1)}$$

d) $M_Z(t) = e^{-\sqrt{\lambda}t + \lambda(e^{t/\sqrt{\lambda}} - 1)}$

$$\ln M_Z(t) = -\sqrt{\lambda}t + \lambda e^{t/\sqrt{\lambda}} - \lambda$$

$$= -\sqrt{\lambda}t - \lambda + \lambda\left[1 + \frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3!\lambda^{3/2}} + \frac{t^4}{4!\lambda^2} + \dots\right]$$

$$= -\sqrt{\lambda}t - \lambda + \left[\lambda + \sqrt{\lambda}t + \frac{t^2}{2} + \frac{t^3}{4!\lambda} + \dots\right]$$

$$= \frac{t^2}{2} + \frac{t^3}{4!\lambda} + \dots$$

$$\lim_{\lambda\to\infty}\left[\ln M_Z(t)\right] = t^2/2$$

$$\Rightarrow M_Z(t) = e^{\frac{1}{2}t^2}$$

d) $\lim_{\lambda \to \infty} \left[-\sqrt{\lambda}\, t + \lambda e^{t/\sqrt{\lambda}} - \lambda \right]$

$= \lim_{\lambda \to \infty} \left\{ \lambda \left[e^{t/\sqrt{\lambda}} - 1 \right] - \sqrt{\lambda}\, t \right\}$

$= \lim_{\lambda \to \infty} \dfrac{e^{t/\sqrt{\lambda}} - 1 - t/\sqrt{\lambda}}{1/\lambda}$

$= \lim_{\lambda \to \infty} \dfrac{e^{t/\sqrt{\lambda}}\, \lambda^{-3/2} t + \frac{1}{2} t \lambda^{-3/2}}{-\lambda^{-2}}$

$= \lim_{\lambda \to \infty} \frac{1}{2} e^{t/\sqrt{\lambda}} \sqrt{\lambda} - \frac{1}{2} t \sqrt{\lambda}$

$= \lim_{\lambda \to \infty} \dfrac{\frac{1}{2} t e^{t/\sqrt{\lambda}} - \frac{1}{2} t}{\lambda^{-1/2}}$

$= \lim_{\lambda \to \infty} \dfrac{\frac{1}{2} t^2 e^{t/\sqrt{\lambda}} \left(-\frac{1}{2} \lambda^{-3/2}\right)}{-\frac{1}{2} \lambda^{-3/2}}$

$= \lim_{\lambda \to \infty} + \frac{1}{2} t^2 e^{t/\sqrt{\lambda}}$

$= \frac{1}{2} t^2$

e) $P[X \geq 22 \text{ or } X \leq 10]$; $\lambda = \mu = 16$

$= P[|X - 16| \geq 6]$

$\leq \dfrac{\sigma_x^2}{C^2}$ $\quad C = 6$

$= \dfrac{(16)}{(6)^2}$

f) $P[X \geq 22] + P[X \leq 10]$

$= P[X_N \geq 21.5] + P[X_N \leq 10.5]$

$= P\left[Z \geq \dfrac{21.5 - 16}{4}\right]$

$+ P\left[Z \leq \dfrac{10.5 - 16}{4}\right]$

$= P\left[Z \geq \dfrac{5.5}{4}\right] + P\left[Z \leq \dfrac{-5.5}{4}\right]$

4-3-73

DEFINITION:

THE GAMMA FUNCTION

$$\Gamma(\alpha) = \int_0^\infty Y^{\alpha-1} e^{-Y}\, dY \qquad \text{FOR } \alpha > 0$$

THEOREM:

$$\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$$

PROOF:

$$\Gamma(\alpha) = \int_0^\infty Y^{\alpha-1} e^{-Y}\, dY$$

LET $u = Y^{\alpha-1}$ $\quad dv = e^{-Y} dY$

$du = (\alpha-1) Y^{\alpha-2} dY, \; v = -e^{-Y}$

$$\Rightarrow \Gamma(\alpha) = -Y^{\alpha-1} e^{-Y}\Big|_0^\infty + (\alpha-1)\int_0^\infty Y^{\alpha-2} e^{-Y}\, dY$$

$$\lim_{Y\to\infty} \frac{Y^{\alpha-1}}{e^{Y}} = 0$$

$$\therefore \Gamma(\alpha) = (\alpha-1)\int_0^\infty Y^{\alpha-2} e^{-Y}\, dY$$
$$= (\alpha-1)\,\Gamma(\alpha-1)$$

COROLLARY

IF α IS A POSITIVE INTEGER

$$\Gamma(\alpha) = (\alpha-1)!$$

PROOF: ABOVE THEM. AND MATHEMATICAL INDUCTION

$\Gamma(1) = 1$, ETC.

THEOREM

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad (= -\tfrac{1}{2}!)$$

PROOF: $\Gamma(\frac{1}{2}) = \int_0^\infty Y^{-1/2} e^{-Y}\, dY$

LET $Y = t^2/2 \Rightarrow dY = t\,dt$; $\quad (t > 0)$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \sqrt{2}\, t^{-1} e^{-t^2/2}\, t\, dt$$
$$= \int_0^\infty \sqrt{2}\, e^{-t^2/2}\, dt$$
$$= \sqrt{2}\sqrt{2\pi}\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \leftarrow \text{N.R.V.}$$
$$= 2\sqrt{\pi}\,(\tfrac{1}{2})$$
$$= \sqrt{\pi}$$

DEFINITION: LET X BE A CONTINUOUS RANDOM VARIABLE WITH DENSITY FUNCTION

$$f_X(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{; FOR } x > 0 \\ 0 & \text{; OTHERWISE} \end{cases}$$

WHERE $\alpha > 0$ AND $\beta > 0$

THEN X IS CALLED A GAMMA RANDOM VARIABLE.

SHOWN

$$\int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)} d\left(x/\beta\right)$$

LET $y = x/\beta$

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y}\,dy$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

FOR $\alpha = 1$ X IS AN EXPONENTIAL RANDOM VARIABLE

$$\Rightarrow f_X(x) = \frac{1}{\beta} e^{-x/\beta} \quad \text{FOR } x > 0$$

THEOREM: IF X IS A Γ RANDOM VARIABLE WITH PARAMETERS α & β, THEN $M_X(t) = (1-\beta t)^{-\alpha}$.

PROOF:

$$M_X(t) = \int_0^\infty e^{xt} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\beta} - t\right)}\,dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{\frac{-x}{\left[\frac{\beta}{1-\beta t}\right]}}\,dx$$

$$= \frac{1}{(1-\beta t)^\alpha} \int_0^\infty \frac{1}{\left(\frac{\beta}{1-\beta t}\right)^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{\frac{-x}{\left(\frac{\beta}{1-\beta t}\right)}}\,dx$$

$$= \frac{1}{(1-\beta t)^\alpha} \int_0^\infty \frac{1}{\beta_1^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta_1}\,dx = \frac{1}{(1-\beta t)^\alpha}$$

COROLLARY: THE MEAN AND VARIANCE OF A Γ RANDOM VARIABLE WITH PARAMETERS ARE $\mu = \alpha\beta$, $\sigma^2 = \alpha\beta^2$

(PROOF LEFT TO STUDENT)

DEFINITION: LET γ BE A POSITIVE INTEGER. A GAMMA RANDOM VARIABLE WITH $\alpha = \gamma/2$ AND $\beta = 2$ IS CALLED A CHI-SQUARE (χ^2) RANDOM VARIABLE WITH γ DEGREES OF FREEDOM. WE USE THE NOTATION χ^2_γ TO REPRESENT A CHI-SQUARED R.V. WITH γ D.O.F.

$$f_{\chi}(x) = \frac{1}{\Gamma(\frac{\gamma}{2})\, 2^{\gamma/2}}\, x^{\frac{\gamma}{2}-1} e^{-x/2} \quad \text{FOR } x > 0$$

$$M_{\chi^2_\gamma}(t) = (1-2t)^{-\gamma/2}$$

$$E[\chi^2_\gamma] = \gamma$$

$$Var[\chi^2_\gamma] = 2\gamma$$

THEOREM: IF Z IS A $N(0,1)$ RANDOM VARIABLE, THE $X = Z^2$ IS A χ^2_1 RANDOM VARIABLE.

4-4-73

STIRLING'S FORMULA

$$\Gamma(x+1) \doteq \sqrt{2\pi}\; x^{x+\frac{1}{2}}\; e^{-x} \quad \text{FOR BIG } x$$

$$M_{\chi^2_r}(t) = (1-2t)^{-r/2}$$

THEOREM: IF Z IS A $N(0,1)$ RANDOM VARIABLE, THEN $X = Z^2$ IS A χ^2_1 RANDOM VARIABLE

(PROBLEM 11, PG 143 IS, FIND M.G.F. OF Z^2 = M.G.F. χ^2_1)

PROOF: $F_X(t) = P[X < t]$

$$= P[Z^2 < t]$$

$$= P[-\sqrt{t} < Z < \sqrt{t}]$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= 2\int_{0}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$f_z(t) = \frac{d}{dt} F_z(t)$$

$$= \frac{d}{dt}\left[\int_0^{g(t)} f(\tau)\, d\tau\right]$$

$$= f\left[g(t)\right] \frac{dg}{dt}$$

$$\Rightarrow f_X(t) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}t} \frac{d}{dt}\sqrt{t}$$

$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}t}\, \frac{1}{2} t^{-1/2}$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}}\, t^{\frac{1}{2}-1} e^{-\frac{1}{2}t}$$

$$= \frac{1}{2^{1/2}\Gamma(1/2)}\, t^{\frac{1}{2}-1} e^{-t/2}$$

A χ^2_1 DISTRIBUTION

THEOREM: LET $X_1, X_2, \ldots, X_n$ BE χ^2 RANDOM VARIABLES (INDEPENDENT) WITH $r_1, r_2, \ldots, r_n$ d.o.f. THEN

$$X = \sum_{i=1}^{n} X_i$$

IS A χ^2_r RANDOM VARIABLE WHERE

$$r = \sum_{i=1}^{n} r_i$$

$$M_X(t) = M_{\Sigma X_i}(t)$$

$$= \prod_{i=1}^{n} (1-2t)^{-r_i/2}$$

$$= (1-2t)^{-\sum_{i=1}^{n} r_i/2}$$

$$= (1-2t)^{-r/2}$$

REMARKS. A χ^2 RANDOM VARIABLE X WITH r D.O.F. CAN BE WRITTEN AS

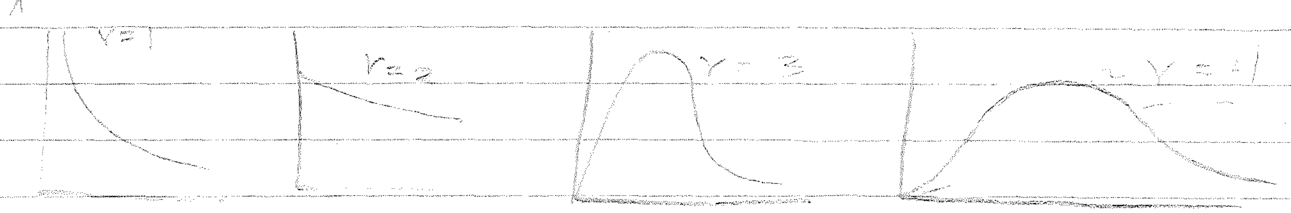
$$X = \sum_{i=1}^{r} X_i \quad \text{WHERE EACH}$$

X_i IS A χ^2_1 RANDOM VARIABLES. THUS $X = \sum_{i=1}^{r} Z_i^2$ WHERE THE Z_i'S ARE INDEPENDENT $N(0,1)$ RANDOM VARIABLES

COROLLARY: LET Y BE A χ^2_r RANDOM VARIABLE & LET $Z_r = (Y - r)/\sqrt{2r}$ & LET $F_r(t)$ BE THE c.d.f. OF Z_r. THEN

$$\lim_{r \to \infty} F_r(t) = \int_{-\infty}^{t} e^{-\frac{1}{2}z^2}\, dz$$

χ^2



<u>THEOREM</u>: LET X_1 AND X_2 BE INDEPENDENT R.V.'S, X_1 BE $\chi^2_{\gamma_1}$ AND

$Y = X_1 + X_2$ BE χ^2_{γ} WITH $\gamma > \gamma_1$. THEN X_2 IS A $\chi^2_{\gamma - \gamma_1}$ RANDOM VARIABLE.

(PROOF LEFT FOR EXERCISE)

<u>THEOREM</u>: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF X WHERE X IS A $N(\mu, \sigma^2)$ R.V.

a) $Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$ IS A χ^2_n R.V.

PROOF IS EASY

b) $\bar{X}$ AND S^2 ARE (BELIEVE IT OR NOT) INDEPENDENT R.V.

PROOF: HOGG & CRAIG ...

(c) $\frac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} R.V.

COMPARE $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \qquad = \chi^2_{n-1}$

TO $\quad Y = \sum_{i=1}^{n} \frac{(X_i - \mu)}{\sigma^2} \qquad = \chi^2_n$

(CONT.)

4-5-73

PROOF OF PART (C):

$$\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^2} = \frac{\sum\left[(x_i-\bar{X})+(\bar{X}-\mu)\right]^2}{\sigma^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{X})^2+\sum_{i=1}^{n}(\bar{X}-\mu)^2+\sum_{i=1}^{n}2(\bar{X}-\mu)(x_i-\bar{X})}{\sigma^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{X})^2}{\sigma^2} + \frac{n(\bar{X}-\mu)^2}{\sigma^2} + \frac{2(\bar{X}-\mu)\sum(x_i-\bar{X})}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} + \left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right]^2 + 0$$

$$\therefore \sum\frac{(x_i-\mu)^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} + \left[\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right]^2$$

$$\Rightarrow \chi^2_n = \frac{(n-1)S^2}{\sigma^2} + Z^2$$

$$= \frac{(n-1)S^2}{\sigma^2} + \chi^2_1$$

$$\Rightarrow \chi^2_{n-1} = \frac{(n-1)S^2}{\sigma^2} = \frac{\sum(x_i-\bar{X})^2}{\sigma^2}$$

THEOREM: LET $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ BE THE ORDER STATISTICS FOR A RANDOM SAMPLE OF SIZE OF n, FROM POP. WITH C.D.F. $F_X(t)$, THEN

$$F_{X_{(n)}}(t) = [F_X(t)]^n \text{ AND}$$

$$F_{X_{(1)}}(t) = 1 - [1 - F_X(t)]^n$$

PROOF

$$F_{X_{(n)}}(t) = P[X_{(n)} < t]$$

$$X_{(n)} > X_{(n-1)} > \ldots > X_{(1)}$$

$$\Rightarrow F_{X_{(n)}}(t) = P[X_1 \le t \text{ AND } X_2 \le t \ldots \text{ AND } X_n \le t]$$

$$= P[X_1 \le t]\,P[X_1 \le t] \ldots P[X \le t]^n$$

$$= [F_X(t)]^n$$

$$F_{X_{(1)}}(t) = P[X_{(1)} \le t]$$

$$= 1 - P[X_{(1)} > t]$$

$$= 1 - P[X_1 > t \text{ AND } X_2 > t \text{ AND} \ldots \text{ AND } X_n > t]$$

$$= 1 - P[X_1 > t]\,P[X_2 > t] \cdot P[X_n > t]$$

$$= 1 - [1 - P(X_1 \le t)][1 - P[X_2 \le t] \cdots [1 - P[X_n \le t)]$$

$$= 1 - [1 - F_X(t)]^n$$

COROLLARY: IF X IS CONTINUOUS WITH DENSITY FUNCTION $f_X(x)$,

$$f_{X_{(n)}}(t) = n\left[F_X(t)\right]^{n-1} f_X(t)$$

AND $$f_{X_{(1)}}(t) = n\left[1 - F_X(t)\right]^{n-1} f_X(t)$$

SUPPOSE $n = 2m+1$ ($m \to$ MEDIAN)

$$f_M(t) = \frac{(2m+1)!}{m!\,m!}\left[F_X(t)\right]^m f_X(t)\left[1 - F_X(t)\right]^m$$

Pg. 274; PROBLEMS 1-9 ANS. ...

4-9-73

Pg 217

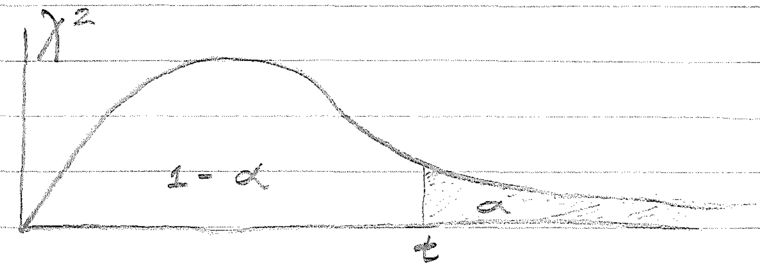
8) a) $P[X_{(1)} = 1]$

$X \; ; \; p(x) \begin{cases} p & \text{FOR } x = 1 \\ 1-p & \text{FOR } x = 0 \end{cases}$

$\Rightarrow P[X_{(1)} = 1] = p^n$

b) $P[X_{(n)} = 1] = 1 - (1-p)^n$ (EH?)

χ^2 TABLE



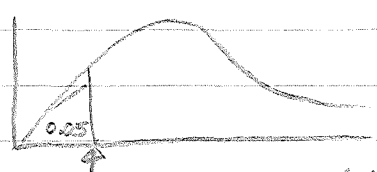
$\Downarrow$

$\chi^2_{r;\alpha}$

a) EX: $r = 6$

$\chi^2_{6;0.05} = 12.6$

b)



$\chi^2_{6;0.95} = 1.64$

3) a) $P\left(\left|\frac{X}{5} - \frac{1}{2}\right| \leq 0.05\right)$

$= P\left[-0.05 < \frac{X}{5} - \frac{1}{2} < 0.05\right]$

$= P\left[0.45 < \frac{X}{5} < 0.55\right]$

$= P[2.25 < X < 2.75] = 0$

b) $P[1.75 < X < 3.25]$

$= P[X = 2] + P[X = 3]$

9) LET

$$F_X(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= P[X \le t] = P\left[Z \le \frac{t-\mu}{\sigma}\right]$$

$$= \int_{-\infty}^{\frac{t-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = N_Z\left(\frac{t-\mu}{\sigma}\right)$$

FIND $f_{X_{(n)}}(t) = n\left[F_X(t)\right]^{n-1} f_X(t)$

$$= n\left[N_Z\left(\frac{t-\mu}{\sigma}\right)\right]^{n-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

FIND $F_{X_{(1)}}(t) = 1 - \left[1 - F_X(t)\right]^n$

$f_{X_{(1)}}(t) = n\left[1 - F_X(t)\right]^{n-1} f_X(t)$

(OMIT 7-1)

SECTION 7-2

DEFINITION: LET X BE A CONTINUOUS R.V. WHOSE DENSITY FUNCTION $f(x;\lambda)$ DEPENDS ON AN UNKNOWN PARAMETER λ. LET $X_1, X_2 \ldots X_n$ BE A RANDOM SAMPLE OF X AND LET $x_1, x_2, \ldots, x_n$ BE THE OBSERVED SAMPLE VALUES. THE LIKELIHOOD FUNCTION OF THE SAMPLE IS

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$$

DEFINITION: LET $L(\lambda)$ BE THE LIKELIHOOD FUNCTION BE THE LIKELIHOOD FUNCTION OF A RANDOM SAMPLE AND $\hat{\lambda}$ BE A VALUE OF λ THAT MAXIMIZES $L(\lambda)$. THEN $\hat{\lambda}$ IS A MAXIMUM LIKELIHOOD ESTIMATE OF λ; AND $\hat{\Lambda}$ THE R.V. WHOSE OUTCOME IS $\hat{\lambda}$ IS A MAX. LIKELIHOOD ESTIMATOR OF λ

4-10-73

REMARK: WE HAVE SIMILAR DEFINITIONS FOR X DISCRETE

EX) $f(x;\mu) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_0}\right)^2}$; σ_0 KNOWN

$$f(x_1, x_2, \ldots, x_n; \lambda) = \prod_{i=1}^{n} f(x_i ; \mu)$$

$$L(\mu) = \prod_{i=1}^{n} f(x_i ; \mu) = \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma_0}\right)^2}$$

LIKELIHOOD FUNCTION (MAXIMUMIZE)

$$\ln L(\mu) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma_0}\right)^2$$

$$\frac{d \ln L(\mu)}{d\mu} = \frac{-1}{2\sigma_0^2}\sum_{i=1}^{n}\frac{d}{d\mu}(x_i-\mu)^2$$

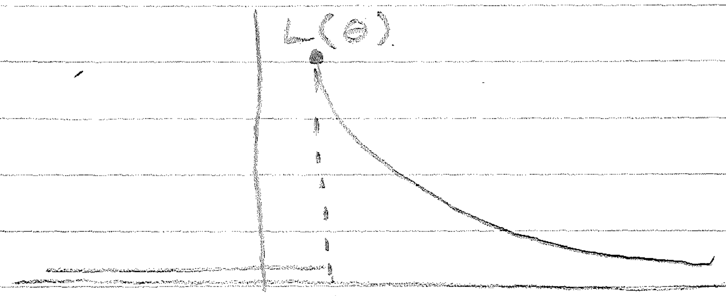
$$= \frac{-1}{2\sigma_0^2}\sum_{i=1}^{n} -2(x_i-\mu)$$

$$= \sum_{i=1}^{n}(x_i-\mu)\Big/\sigma_0^2 = 0$$

$$\Rightarrow \mu = \sum_{i=1}^{n} x_i \Big/ n = \overline{X} \leftarrow \text{MAXIMUM LIKELIHOOD ESTIMATE}$$

EX) $f(x;\theta) = \begin{cases} \frac{1}{\theta} & ; 0 \le x \le \theta \\ 0 & ; \text{OTHERWISE} \end{cases}$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \begin{cases} \frac{1}{\theta^n} & ; 0 \le x_i \le \theta \\ 0 & ; \text{OTHERWISE} \end{cases}$$



$$\Rightarrow L(\theta) = \begin{cases} \frac{1}{\theta^n} & \text{FOR } \theta \ge X_{(n)} \\ & \text{OTHERWISE} \end{cases}$$

$\Rightarrow$ MAXIMUM LIKELIHOOD ESTIMATER FOR θ IS $X_{(n)}$

MAXIMIZING TWO PARAMETERS (μ & V)

$$L(\mu, V) = \prod_{i=1}^{n} f[x_i; \mu, V]$$

$$= \left(\frac{1}{2\pi V}\right)^{n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{V}}$$

$$\ln(L(\mu,V)) = -\frac{n}{2}\ln 2\pi V - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{V}\right)^2$$

$$\frac{\delta \ln L(\mu,V)}{\delta \mu} = -\frac{1}{2V}\sum_{i=1}^{n} -2(x_i-\mu) = \frac{1}{V}\sum_{i=1}^{n}(x_i-\mu)$$

$$\frac{\delta \ln L(\mu,V)}{\delta V} = -\frac{n}{2V} + \frac{1}{2V^2}\sum (x_i-\mu)^2$$

SET TO ZERO

$$\frac{1}{V}\sum_{i=1}^{n}(x_i-n) = 0 \Longrightarrow \mu = \bar{x}$$

$$-\frac{n}{2V} + \frac{1}{2V^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 = 0$$

$$\Longrightarrow V = \frac{1}{n}\sum (x_i-\bar{x})^2$$

DEFINITION: LET Γ BE A POINT ESTIMATER OF THE UNKNOWN POPULATION PARAMETER γ. Γ IS SAID TO BE AN UNBIASED ESTIMATOR OF γ IS $E[\Gamma] = \gamma$

THEOREM: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF A RANDOM VARIABLE X, WHERE X HAS MEAN μ AND VARIANCE σ^2. THEN

a) $\bar{X}$ IS AN UNBIASED ESTIMATOR OF μ

b) S^2 IS " " " OF σ^2

4-11-73

PROOF OF (b)

$$E[S^2] = E\left[\frac{\sum X_i^2 - n\bar{X}^2}{n-1}\right]$$

$$= \frac{\sum_{i=1}^{n} E(X_i^2) - nE[\bar{X}^2]}{n-1}$$

RECALL $E[\bar{X}^2] = \sigma_{\bar{X}}^2 + \mu_{\bar{X}}^2 = \frac{\sigma^2}{n} + \mu^2$

RECALL $\sigma_x^2 = E[X^2] - \mu_x^2 \Rightarrow E[X^2] = \sigma_x^2 + \mu_x^2$

$$\Rightarrow E(S^2) = \frac{\sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)}{n-1}$$

$$= \frac{n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2}{n-1}$$

$$= \sigma^2$$

THEN $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n} = \frac{n-1}{n} S^2$

AND $E[\hat{\sigma}^2] = \frac{n-1}{n}\sigma^2 = \sigma^2 - \frac{1}{n}\sigma^2$

c) LET X BE A BINOMIAL R.V WITH PARAMETERS OF n & p. THEN X/n IS AN UNBIASED ESTIMATOR OF p.

DEFINITION: IF Γ_1 & Γ_2 ARE BOTH UNBIASED ESTIMATORS OF THE POPULATION PARAMETE γ BASED ON THE SAME RANDOM SAMPLE, THEN Γ_1 IS MORE EFFICIENT THAN Γ_2 IF

$$Var(\Gamma_1) < Var(\Gamma_2)$$

SUPPOSE AS AN EXAMPLE WE TAKE A SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION. THE $\bar{x}$ AND M_o (MEDIAN) ARE UNBIASED ESTIMATORS OF μ.

$$V(\bar{x}) = \sigma^2/n$$

FOR n LARGE

$$V(M_o) \simeq \frac{\pi}{2}\frac{\sigma^2}{n}$$

$$\left(Var\left[\frac{x_1 + x_n}{2}\right] = \frac{\sigma^2}{2} \right)$$

THEOREM: (CRAMER-RAO INEQUALITY)

UNDER CERTAIN REGULARITY CONDITIONS (AMONG WHICH IS THE CONDITION THE DOMAIN OF POSITIVE VALUES OF DENSITY FUNCTION DOES NOT INCLUDE γ). IF Γ IS AN UNBIASED ESTIMATOR OF γ, THEN

$$Var(\Gamma) \geq \frac{1}{n E\left[\left\{\frac{\partial}{\partial\gamma} \ln f(X;\gamma)\right\}^2\right]}$$

WHERE f IS THE DENSITY FUNCTION OF THE CONTINUOUS POPULATION R.V. X, THOUGHT OF AS A FUNCTION OF X AND γ (A SIMILAR STATEMENT HOLDS FOR X DISCRETE)

REMARK: THIS MEANS THAT IF

$$Var(\Gamma) = \frac{1}{n E\left[\left\{\frac{\partial}{\partial\gamma} \ln f(X;\gamma)\right\}^2\right]}$$, THEN THERE DOES NOT EXIST A MORE EFFICIENT UNBIASED ESTIMATOR OF γ

4-13-73

TEST WEDNESDAY

SEC 6.3, 7.2, 7.3 OMIT 7.1

1-9 - Pg 236-7

LOWER LIMIT OF INEQUALITY

$$\Rightarrow \frac{1}{nE\left[\left\{\frac{\delta}{\delta\gamma} \ln f(X;\gamma)\right\}^2\right]}$$

SUPPOSE WE TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma_0^2)$ POPULATION, σ_0^2 IS KNOWN. WE WANNA MAKE A POINT ESTIMATOR FOR μ WHICH IS UNBIASED

$$f(X;\mu) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma_0}\right)^2}$$

$$\ln f(X;\mu) = -\ln \sqrt{2\pi}\,\sigma_0 - \frac{1}{2}\left(\frac{X-\mu}{\sigma_0}\right)^2$$

$$\frac{\delta \ln f(X;\mu)}{\delta\mu} = \frac{X-\mu}{\sigma_0^2}$$

$$\left[\frac{\delta \ln f(X;\mu)}{\delta\mu}\right]^2 = \frac{(X-\mu)^2}{\sigma_0^4}$$

$$E\left[\left\{\frac{\delta}{\delta\mu} \ln f(X;\mu)\right\}^2\right] = \frac{E\left[(X-\mu)^2\right]}{\sigma_0^4} = \frac{1}{\sigma_0^2}$$

$$\Rightarrow \frac{1}{nE\left[\left(\frac{\delta}{\delta\gamma} \ln f(X;\gamma)\right)^2\right]} = \frac{\sigma_0^2}{n}$$

THE LOWER BOUND FOR CRAMER-RAO INEQUALITY. SINCE

$$VAR(\bar{X}) = \frac{\sigma_0^2}{n}$$

SO THAT $\bar{X}$ IS THE BEST

DEFINITION: Γ IS A CONSISTANT ESTIMATOR OF γ IF

$$\lim_{n\to\infty} P[|\Gamma - \gamma| > \epsilon) = 0 \quad \forall \epsilon > 0$$

THEOREM: IF $\lim_{n\to\infty} E[\Gamma] = \gamma$ AND $\lim_{n\to\infty} VAR(\Gamma) = 0$, THEN Γ IS A CONSISTENT ESTIMATOR OF γ.

PROOF FOR $E[\Gamma] = \gamma$ ALWAYS

CHEBYCHEV'S INEQUALITY SAYS

$$P(|\Gamma - \gamma| \geq \epsilon) \leq \frac{1}{\epsilon^2} VAR(\Gamma)$$

$$0 \leq \lim_{n\to\infty} P(|\Gamma - \gamma| \geq \epsilon) \leq \lim_{n\to\infty} \frac{1}{\epsilon^2} VAR(\Gamma)$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} P[|\Gamma - \gamma| \geq \epsilon] \overset{\leq 0}{\leq} 0$$

OR $\lim_{n\to\infty} P[|\Gamma - \gamma| \geq \epsilon] = 0$

CHEBYCHEV'S INEQUALITY GENERALIZATION

$$P[|X - c| \geq \epsilon] \leq \frac{1}{\epsilon^2} E[(x-c)^2]$$

OBVIOUSLY, IF WE TAKE A RANDOM SAMPLE FROM ANY POPULATION, $\overline{X}$ IS CONSISTANT ESTIMATOR OF μ, WHEN $\overline{X}$ HAS A FINITE VARIANCE.

SUPPOSE WE TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION, WE KNOW S^2 IS AN UNBIASED ESTIMATOR OF σ^2 FOR ALL n. ALSO

$\frac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} R.V.

LET'S GANDER AT

$$Var\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1)$$

RECALL $Var[cX] = c^2 Var[X]$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} Var(S^2) = 2n-1$$

$$Var(S^2) = \frac{2\sigma^4}{n-1}$$

$$\lim_{n\to\infty} Var(S^2) = 0$$

IF $E[X^2] = a$ IS IT TRUE THAT $E[X] = \sqrt{a^2}$ (NO!)

NOTE:

$E(S) \neq \sigma$ BUT $E(S) = C_n \sigma \ni \lim_{n\to\infty} C_n = 1$

S IS CONSISTANT FOR σ

IN QUALITY CONTROL

$E[d_n R] = \sigma \ni R = \text{RANGE}$

GOOD ONLY FOR SMALL n SINCE

$d_n R$ IS NOT CONSISTENT FOR σ

CHECK OUT THEOREM 7.3.3 AND

THE PARAGRAPH ABOVE

4-16-73

EXAM ON THURSDAY ~~~~ arg! Nof? Lot

Pg 236

1) $C[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2] = U$

$X_i \sim N(\mu, \sigma^2)$

$X_1 - X_2 \sim N(0, 2\sigma^2)$

$U = 2\sigma^2 C\left[\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_3 - X_4}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_5 - X_6}{\sqrt{2}\sigma}\right)^2\right]$

$\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right) \sim N(0,1)$

$U = 2\sigma^2 C \chi^2_3$

$E[2\sigma^2 C\chi^2_3] = 2\sigma^2 C E[\chi^2_3]$ $\qquad [E[\chi^2_\nu] = \nu]$

$= 6\sigma^2 C$

$\Rightarrow E[U] = 6C\sigma^2$

WE WANT $U \ni$

$E[U] = \sigma^2 \Rightarrow C = \frac{1}{6}$

2) $Var(U) = Var(2\sigma^2 C\chi^2_3)$

$= Var\left(\frac{\sigma^2}{3}\chi^2_3\right)$

$= \frac{\sigma^4}{9} Var(\chi^2_3)$; $Var\,\chi^2_\nu = 2\nu$

$\Rightarrow Var(U) = \frac{2}{3}\sigma^4$

$Var(s^2) = \frac{2\sigma^4}{n-1} = \frac{2}{5}\sigma^4$

s^2 IS THE MORE EFFICIENT ESTIMATOR

IN QUALITY CONTROL

$E[d_n R] = \sigma \ni R = \text{RANGE}$

GOOD ONLY FOR SMALL n SINCE

$d_n R$ IS NOT CONSISTENT FOR σ

CHECK OUT THEOREM 7.3.3 AND THE PARAGRAPH ABOVE

4-16-73

EXAM ON THURSDAY — Avg. Max? Lot

Pg 236

1) $C\left[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2\right] = U$

$X_i \sim N(\mu, \sigma^2)$

$X_1 - X_2 \sim N(0, 2\sigma^2)$

$U = 2\sigma^2 C\left[\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_3 - X_4}{\sqrt{2}\sigma}\right)^2 + \left(\frac{X_5 - X_6}{\sqrt{2}\sigma}\right)^2\right]$

$\left(\frac{X_1 - X_2}{\sqrt{2}\sigma}\right) \sim N(0,1)$

$U = 2\sigma^2 C \chi^2_3$

$E[2\sigma^2 C \chi^2_3] = 2\sigma^2 C E[\chi^2_3]$ $\qquad [E[\chi^2_\nu] = \nu]$

$= 6\sigma^2 C$

$\Rightarrow E[U] = 6C\sigma^2$

WE WANT $U \ni$

$E[U] = \sigma^2 \Rightarrow C = \frac{1}{6}$

2) $Var(U) = Var(2\sigma^2 C \chi^2_3)$

$= Var\left(\frac{\sigma^2}{3}\chi^2_3\right)$

$= \frac{\sigma^4}{9} Var(\chi^2_3)$; $Var\, \chi^2_\nu = 2\nu$

$\Rightarrow Var(U) = \frac{2}{3}\sigma^4$

$Var(s^2) = \frac{2\sigma^4}{n-1} = \frac{2}{5}\sigma^4$

S^2 IS THE MORE EFFICIENT ESTIMATOR

4) $U_n = \frac{1}{n} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2$

$= 2\sigma^2 \left(\frac{1}{n}\right) \chi^2_{n/2}$

$Var(U_n) = \frac{4}{n^2} \sigma^4 n = \frac{4\sigma^4}{n}$

3) $\overline{X} \sim N(\mu, \sigma^2)$

$Y = a\overline{X_1} + (1-a)\overline{X_2}$

$E[Y] = a\mu + (1-a)\mu = \mu$

$var\ Y = a^2 \, var\, \overline{X_1} + (1-a)^2 Var(\overline{X_1})$

$= a^2 \frac{\sigma^2}{\overline{X_1}} + (1-a)^2 Var(\overline{X_1})$

$V = a^2 \frac{\sigma^2}{n_1} + (1-a^2) \frac{\sigma^2}{n_2}$

$\frac{\delta V}{\delta a} = \frac{2a\sigma^2}{n_1} - 2(1-a)\sigma^2/n_2 = 0$

$\left(\frac{2\sigma^2}{n_1} + \frac{2\sigma^2}{n}\right) a - \frac{2\sigma^2}{n_2} = 0$

$\left(\frac{1}{n_1} + \frac{1}{n_2}\right) a = 1/n_2$

$\Rightarrow a = \frac{n_1 n_2^2}{n_1 + n_2} \quad a = \frac{1}{n_2}$

$\Rightarrow a = \frac{n_1}{n_1 + n_2}$

$\Sigma = \frac{n_1 \overline{Y_1}}{n_1 + n_2} + \frac{n_2}{n_1 + n_2} \overline{X_2}$

Pg 236

8) $\Gamma_1 = X_{(n)}$

$\Gamma_2 = 2\bar{X}$

CRAMER RAO

χ^2 ; MAXIMUM DENSITY FUNC.

CONSISTANCY

$\mu_x = \theta/2$

$\sigma_x^2 = \frac{\theta^2}{12}$

$E[\Gamma_2] = 2E[\bar{X}] = \theta$

$Var[\Gamma_2] = 4\,Var[\bar{X}] = 4\,\frac{\sigma_x^2}{n} = \frac{\theta^2}{3n}$

$f_{X_{(n)}}(t) = n[F_x(t)]^{n-1} f_x(t)$

$= n\left(\frac{t}{\theta}\right)^{n-1}\frac{1}{\theta} \quad$ FOR $\quad 0 \le t \le \theta$

$$f_{X_{(n)}}(t) = \begin{cases} \frac{n}{\theta^n}\, t^{n-1} & 0 \le t \le \theta \\ 0 & \text{OTHERWISE} \end{cases}$$

$\Rightarrow E_{X_{(n)}}(t) = \int_0^\theta t f_x(t)\,dt$

$= \frac{n}{\theta^n}\int_0^\theta t^n\,dt$

$= \frac{n}{\theta^n}\,\frac{\theta^{n+1}}{n+1} = \frac{n\theta}{n+1}$

$\sigma_{X_{(n)}}^2 = E[X_{(n)}^2] - \mu_x^2$

$E[X_{(n)}^2] = \int_0^\theta t\, f_{X_{(n)}}\, t\,dt$

$= \frac{n}{\theta^n}\int_0^\theta t^{n+1}\,dt = \frac{n}{\theta^n}\,\frac{\theta^{n+2}}{n+2}$

$= \frac{n}{n+2}\theta^2$

$Var(X_n) = \frac{n}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\theta^2$

$= \frac{n^3+2n^2+n-n^3-2n^2}{(n+2)(n+1)^2}\theta^2$

$= \frac{n}{(n+1)(n+2)^2}\theta^2$

THIS IS A BETTER ESTIMATOR

4-15-73 (OUT OF SEQUENCE)

THEOREM: IF X IS A CONTINUOUS R.V. WITH DENT CITY FUNCTION $f(x;\gamma)$ WHICH DEPENDS ON AN UNKNOWN PARAMETER γ, AND $\hat{\Gamma}$ IS A MAXIMUM LIKELIHOOD ESTIMATER OF γ BASED ON A RANDOM SAMPLE SIZE n OF X, THEN UNDER CERTAIN REGULARITY CONDITIONS (INCLUDING REGION OF THE POSITIVE DENSITY DOES NOT DEPEND ON γ) $\hat{\Gamma}$ IS FOR n LARGE APPROXIMATELY A NORMAL RANDOM VARIABLE WITH MEAN γ AND VARIANCE

$$\frac{1}{nE\left[\left\{\frac{\partial}{\partial\gamma}\ln f(x;\gamma)\right\}^2\right]}$$

A SIMILAR STATEMENT HOLDS FOR X DISCRETE. (NO PROOF)

REMARK

1) $\hat{\Gamma}$ IS ASYMTOTICALLY UNBIASED
2) $\hat{\Gamma}$ IS CONSISTANT
3) $\hat{\Gamma}$ IS THE MOST EFFICIENT ESTIMATOR FOR LARGE n

THEOREM: (INVARIANCE PRINCIPLE OF MAXIMUM LIKELIHOOD)

LET $\hat{\Gamma}$ BE A MAXIMUM LIKELIHOOD ESTIMATOR OF γ. LET $\Theta = h(\gamma)$. THEN $\hat{\Theta} = h[\hat{\Gamma}]$ IS A MAXIMUM LIKELIHOOD ESTIMATOR OF Θ.

(NO PROOF)

4-17-73

Pg 236

7) $Y = \frac{1}{\bar{X}}$ IS THE M.L.E. OF λ FOR $f(x) = \lambda e^{-\lambda x} \mu(x)$

THE SAMPLE IS A NORMAL DISTRIBUTION WITH $\mu = \lambda$ AND σ^2 IS THE LOWER BOUND OF CRAMER-RAO INEQUALITY

5) $P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$ FOR $X = 0, 1, 2, \ldots$

MAX. LIKELIHOOD ESTIMATER OF λ IS $\bar{X}$ $(= \hat{\lambda})$

$\bar{X} = \frac{279}{310}$

$P(X=0) = e^{-\lambda}$

SO MAXIMUM LIKELIHOOD ESTIMATER OF $P(X=0)$ IS $e^{-\bar{X}}$

SHOW THAT FOR ALL n

$P(X) = \frac{e^{-\lambda}\lambda^x}{x!}$, $\bar{X}$ IS THE M.L.E. OF λ

$Var(\bar{X}) = \frac{\sigma_x^2}{n} = \frac{\lambda}{n}$

Pg 237

9) $f(x) = \frac{1}{\mu} e^{-x/\mu}$ FOR $x > 0$

ESTIMATE $\mu \; (= 1/\lambda)$

$f(x, \mu) = \frac{1}{\mu} e^{-x/\mu}$

$f_{X_{(1)}}(t) = n\left[1 - F_X(t)\right]^{n-1} f_X(t)$

$F_X(t) = P[X \le t] = \int_0^t \frac{1}{\mu} e^{-x/\mu} dx$

$= -e^{-x/\mu} \Big|_0^t = 1 - e^{-t/\mu}$

$1 - F_X(t) = e^{-t/\mu}$

$\Rightarrow f_{X_{(1)}}(t) = n\left(e^{-t/\mu}\right)^{n-1} \left(\frac{1}{\mu} e^{-t/\mu}\right)$

$= \frac{n}{\mu} e^{-\frac{nt}{\mu}}$ FOR $t \ge 0$

$E[X_{1}] = \int_0^\infty \frac{nt}{\mu} e^{-\frac{nt}{\mu}} dt = \frac{\mu}{n}$

LET $Y = n X_{(1)} \Rightarrow E[Y] = \mu$

$Var(\bar{X}) = \frac{\sigma_x^2}{n} = \frac{\mu^2}{n}$

$Var(X_{(1)}) = \frac{\mu^2}{n^2}$

$Var(Y) = Var(n X_{(1)}) = \mu^2$

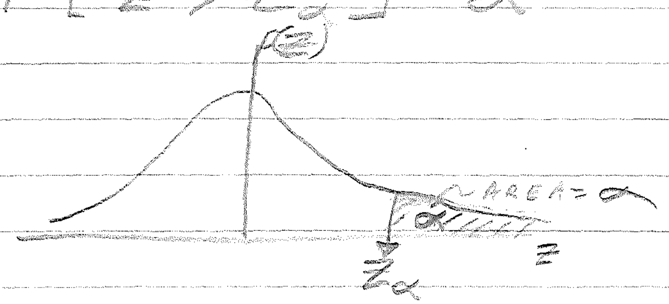
4-23-73 NEW STUFF

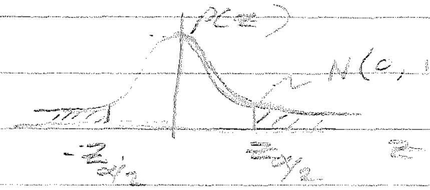
CONFIDENCE INTERVALS

DEFINITION: LET X BE A RANDOM VARIABLE WITH PROBIBILITY OR DENSITY FUNCTION DEPENDING ON AN UNKNOWN PARAMETER θ. LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF X; AND LET L_1 AND L_2 BE STATISTICS BASED ON THE RANDOM SAMPLE. THEN L_1 AND L_2 FORM A $1-\alpha$ CONFIDENCE INTERVAL IF $P[L_1 \le \theta \le L_2] = 1-\alpha$.

(SUPPOSE WE TAKE A SAMPLE OF n FROM A $N(\mu, \sigma_0^2)$ POPULATION, WITH σ_0^2 KNOWN.) EH?

LET Z BE THE $N(0,1)$ RANDOM VARIABLE. Z_α IS A NUMBER SUCH THAT $P[Z > Z_\alpha] = \alpha$





LET α BE A SMALL POSITIVE NUMBER

$$P(-z_{\alpha/2} \le z \le z_{\alpha/2}) = 1-\alpha$$

NOW

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\Rightarrow \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z \sim N(0,1)$$

$$\therefore P\left[-z_{\alpha/2} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right] = 1-\alpha$$

$$= P\left[-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{X}-\mu \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

$$= P\left[-\bar{X}-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le -\mu \le -\bar{X}+z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] = 1-\alpha$$

$$= P\left[\bar{X}+z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \ge \mu \ge \bar{X}-z_{\alpha/2}\sigma/\sqrt{n}\right] = 1-\alpha$$

$$= P\left[\bar{X}-z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{X}+z_{\alpha/2}\sigma/\sqrt{n}\right] = 1-\alpha$$

$$L_1 = \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad ; \quad L_2 = \bar{X} + z_{\alpha/2}\,\sigma/\sqrt{n}$$

HOT DAMN!

4-24-73

TAKE A SAMPLE FROM A $N(\mu, \sigma_0^2)$ R.V.

$$\Rightarrow P[\overline{X} - Z_{\alpha/2}\,\sigma_0/\sqrt{n} \le \mu \le \overline{X} + Z_{\alpha/2}\,\sigma_0/\sqrt{n}] = 1-\alpha$$

WHERE $P[Z > Z_{\alpha/2}] = \alpha/2$

$$L_1 = \overline{X} - Z_{\alpha/2}\,\sigma_0/\sqrt{n}$$

$$L_2 = \overline{X} + Z_{\alpha/2}\,\sigma_0/\sqrt{n}$$

FOR LARGE n AND UNKNOWN σ_0^2

$$L_1 = \overline{X} - Z_{\alpha/2}\sqrt{S^2}/\sqrt{n}$$

$$L_2 = \overline{X} + Z_{\alpha/2}\sqrt{S^2}/\sqrt{n}$$

DEFN: LET Z BE A STANDARDIZED NORMAL RANDOM VARIABLE AND V BE AN INDEPENDENT χ^2_r R.V., THEN

$$T = \frac{Z}{\sqrt{V/r}}$$ IS A STUDENT'S t R.V. WITH r DEGREES OF FREEDOM, AND IS DENOTED BY T_r

THEOREM: SUPPOSE WE TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION. THEN $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ IS A T_{n-1} R.V.

PROOF

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} = \frac{Z}{\sqrt{\frac{\chi^2_{n-1}}{n-1}}} = T_{n-1}$$

THEOREM: THE DENSITY FUNCTION OF A T_r IS

$$f_{T_r}(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\sqrt{r\pi}}\left[1 - \frac{t^2}{r}\right]^{-\left(\frac{r+1}{2}\right)}$$

PROOFS:

Pp. 201-2 OF FREUND'S MATHEMATICAL STATISTICS

HOGG & CRAIG'S MATH. STATISTICS (BETTER)

RECALL

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim T_{n-1} \; ; \; \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim Z \Longrightarrow$$

THEOREM:

$$\lim_{r \to \infty} f_{T_r}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \; ; \; -\infty < t < \infty$$

PROOF LEFT FOR STUDENTS

(HINTS)

STERLING'S FORMULA

a) $$\lim_{x \to \infty} \frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+1/2} e^{-x}} = 1$$ (FULK'S ADV. CALC. pg 461)

b) $$\lim_{x \to \infty} \frac{\Gamma(a+x)}{\Gamma(b+x)} = x^{a-b}$$ (" " " pg 462)

c) $$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

4-2?-73

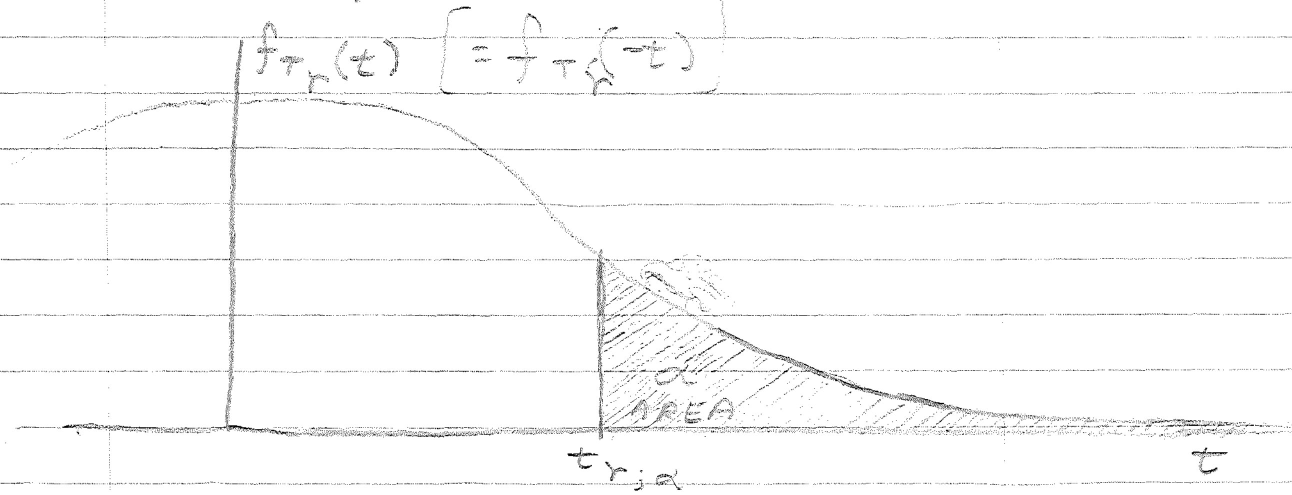
$$\frac{\Gamma(a+x)}{\Gamma(b+x)} \to x^{a-b} \quad \text{AS } x \to \infty$$

FROM $\Gamma(a+x) \to \sqrt{2\pi}\, x^{x+a-1/2} e^{-x}$ AS $x \to \infty$

AND $\Gamma(a+x) = \Gamma(x+a-1+1)$

RECALL

$$f_{T_r}(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}\right)\sqrt{r\pi}} \left(1 + \frac{t^2}{r}\right)^{-\left(\frac{r+1}{2}\right)} \quad \text{FOR } -\infty < t < \infty$$



DEFN: $t_{r;\alpha}$ IS A NUMBER SUCH THAT

$$P[T_r > t_{r;\alpha}] = \alpha$$

EX) $t_{10;0.01} = 2.764$; $t_{5;.05} = 2.015$

IF WE TAKE A SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION, THEN $\frac{\bar{X}-\mu}{S/\sqrt{n}} : T_{n-1}$

$$(-t_{n-1;\alpha/2} \le T_{n-1} \le t_{n-1;\alpha/2}) = 1-\alpha$$

$$P\left[-t_{n-1;\alpha/2} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{n-1;\alpha/2}\right) = 1-\alpha$$

$$= P\left[\bar{X} - t_{n-1;\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{n-1;\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

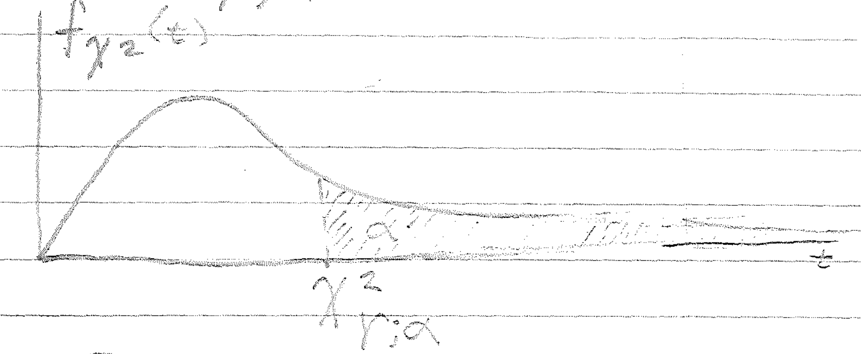
$$L_1 = \bar{X} - t_{n-1;\alpha/2}\, S/\sqrt{n}$$

$$L_2 = \bar{X} - t_{n-1;\alpha/2}\, S/\sqrt{2}$$

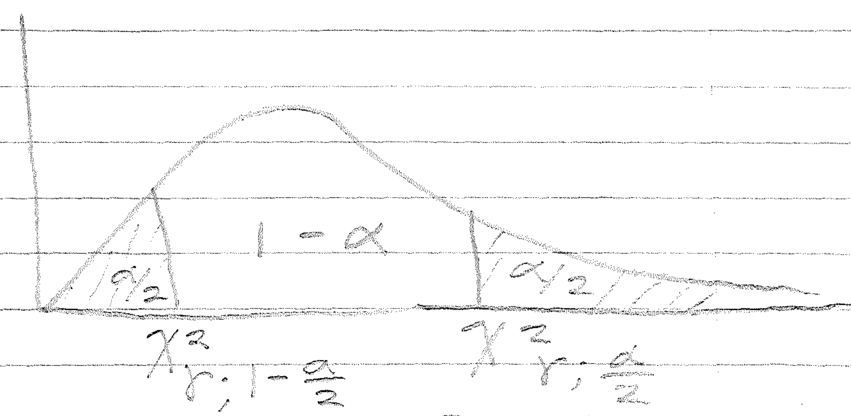
II.

DEFN: $\chi^2_{r;\alpha}$ IS A NUMBER SUCH THAT

$$P[\chi^2_r > \chi^2_{r;\alpha}) = \alpha$$



CONSIDER:



$$P\left[\chi^2_{r;1-\frac{\alpha}{2}} \le \chi^2_r \le \chi^2_{r;\alpha/2}\right] = 1-\alpha$$

IF WE TAKE A RANDOM SAMPLE OF n FROM A $N(\mu, \sigma^2)$ POPULATION

$\frac{(n-1)s^2}{\sigma^2}$ IS A χ^2_{n-1} R.V.

NOW

$$P\left[\chi^2_{n-1;1-\frac{\alpha}{2}} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{n-1;\frac{\alpha}{2}}\right] = 1-\alpha$$

$$= P\left[\frac{1}{\chi^2_{n-1,\alpha/2}} \le \frac{\sigma^2}{(n-1)s^2} \le \frac{1}{\chi^2_{n-1;1-\frac{\alpha}{2}}}\right]$$

$$= P\left[\underbrace{\frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}}_{L_1} \le \sigma^2 \le \underbrace{\frac{(n-1)s^2}{\chi^2_{n-1;1-\alpha/2}}}_{L_2}\right]$$

TO FIND INTERVAL FOR STANDARD DEVIATION, TAKE SQUARE ROOT

4-30-73

a) SAMPLE FROM A $N(\mu, \sigma^2)$ POP., WITH σ^2 KNOWN

FOR μ

$\bar{X} \pm Z_{\alpha/2}\,\sigma_0/\sqrt{n}$ IS A $1-\alpha$ CONF. INTERVAL

b) SAMPLE FROM A $N(\mu, \sigma^2)$ POP., σ^2 NOT KNOWN

FOR μ

$\bar{X} \pm t_{n-1;\alpha/2}\, s/\sqrt{n}$ IS A $1-\alpha$ CONF INTER

c) TAKE A SAMPLE FROM A $N(\mu, \sigma^2)$ POP.

WITH σ^2 NOT KNOWN

$\dfrac{(n-1)s^2}{\chi^2_{n-1;\alpha/2}}, \dfrac{(n-1)s^2}{\chi^2_{n-1;1-\alpha/2}}$ IS A $1-\alpha$

CONF. INTERVAL FOR σ^2

$P\left[-Z_{\alpha/2} \le Z \le Z_{\alpha/2}\right] = 1-\alpha$

FOR LARGE SAMPLE, np FROM BINOMIAL R.V.

$P\left[-Z_{\alpha/2} \le \dfrac{X-np}{\sqrt{np(1-p)}} \le Z_{\alpha/2}\right] \doteq 1-\alpha$

LET $\hat{p} = X/n$

(IN BOOK) CONF. INT. IS

$\hat{p} \pm Z_{\alpha/2}\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

THEM, LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF X WHERE

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \forall\, x \geq 0 \\ 0 & \text{OTHERWISE} \end{cases}$$

THEN: $2\lambda n\bar{X}$ IS A χ^2_{2n} R.V.

PROOF:

$$M_X(t) = \frac{1}{(1 - t/\lambda)}$$

$\left\{ \begin{array}{l} \text{SINCE } M_X(t) = (1-\beta t)^{-\alpha} \text{ FOR A } \Gamma \text{ R.V.} \\ \text{AND } \beta = 1/\lambda \text{ AND } \alpha = 1 \end{array} \right\}$

$$M_{\Sigma X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$= \frac{1}{(1 - t/\lambda)^n}$$

$$M_{2\lambda n\bar{X}}(t) = M_{2\lambda \Sigma X_i}(t) = \frac{1}{\left(1 - \frac{2\lambda t}{\lambda}\right)^n}$$

$$= (1 - 2t)^{-n}$$

$$= (1 - 2t)^{-\frac{(2n)}{2}}$$

WHICH IS THE MOMENT GENERATING FUNCTION OF A χ^2_{2n} R.V.

$$P\left[\chi^2_{2n;1-\alpha/2} \le \chi^2_{2n} \le \chi^2_{2n,\alpha/2}\right] = 1-\alpha$$

$$= P\left[\chi^2_{2n;1-\alpha/2} \le 2n\lambda\bar{x} \le \chi^2_{2n,\alpha/2}\right] = 1-\alpha$$

$$= P\left[\frac{\chi^2_{2n;1-\alpha/2}}{2n\bar{x}} \le \lambda \le \frac{\chi^2_{2n;\alpha/2}}{2n\bar{x}}\right] = 1-\alpha$$

$\Rightarrow 1-\alpha$ CONF INTERVAL FOR λ IN AN EXPON. R.V.

SUPPOSE $\hat{\Theta}$ IS THE M.L.E. OF Θ FOR n LARGE, $\hat{\Theta}$ IS APPROX. A $N(\Theta, V)$ WHERE

$$V = nE\left[\left\{\frac{\partial}{\partial\Theta}\ln f(x;\Theta)\right\}^2\right]$$

SO THAT GENERALLY

$$P\left[-z_{\alpha/2} \le Z \le z_{\alpha/2}\right] = 1-\alpha$$

$$P\left[-z_{\alpha/2} \le \frac{\hat{\Theta}-\Theta}{\sqrt{V}} \le z_{\alpha/2}\right]$$

PROBLEMS

PG
250-1 ALL BUT 13 in C... in TEXT
IN FREUND "MATHEMATICAL STATISTICS" BOOK

PG. 233; #2
198; #9
235; DERIVE STATED APPROXIMATE DISTRIBUTION OF S STATED IN PROBLEM 4 AND DERIVE THE CONF. INTERVAL IN PROB 4
USE THE RESULT OF PROB. #4 TO DO #5

5-1-73

TEST OF HYPOTHESIS (CHAPT. 7 PROB. QUESTIONS TOMORROW)

EXAMPLE

$H_0: \mu = 63''$ (AVE. HT. OF ST. [illegible])

$H_1: \mu > 63''$

WE KNOW $\sigma = 2$ AND POP. IS $N(\mu, \sigma^2)$

LET $n = 16$ BE THE NUMBER OF SAMPLES

$\begin{cases} H_0 \text{ IS THE NULL HYPOTHESIS} \\ H_1 \text{ IS THE ALTERNATIVE HYPOTHESIS} \end{cases}$

$\begin{cases} \text{ACCEPT } H_0 \text{ IF } \bar{X} \le K \\ \text{REJECT } H_0 \text{ IF } \bar{X} > K \end{cases}$

DEFINITION: A HYPOTHESIS IS A STATEMENT ABOUT THE DENSITY (OR PROBABILITY) FUNCTION OF A RANDOM VARIABLE.

DEFINITION: IF THE HYPOTHESIS SPECIFIES COMPLETELY THE DENSITY (OR PROBABILITY) FUNCTION OF THE RANDOM VARIABLE (INCLUDING THE VALUES OF ALL PARAMETERS), IT IS CALLED A SIMPLE HYPOTHESIS. OTHERWISE IT IS A COMPOSITE HYPOTHESIS.

(THE ABOVE EXAMPLE HAS H_0 AS A SIMPLE HYPOTHESIS AND H_1 IS COMPOSITE. IF σ WAS NOT KNOWN, BOTH H_0 AND H_1 ARE COMPOSITE HYPOTHESIS)

<u>DEFINITION</u>: SUPPOSE WE TAKE A
RANDOM SAMPLE $X_1, X_2, \ldots, X_n$ OF
THE RANDOM VARIABLE X AND LET
[illegible]
HYPOTHESIS. THE [illegible] TEST OF
THIS HYPOTHESIS IS A PARTITIONING
OF THE SAMPLE SPACE FOR THE
EXPERIMENT OF TAKING THE RANDOM
SAMPLE, INTO TWO PARTS
1) THE REJECTION (OR CRITICAL) REGION
2) THE ACCEPTANCE REGION.
WE REJECT THE HYPOTHESIS IF
THE n-TUPLE OF OUTCOMES
$x_1, x_2, \ldots, x_n$ FALLS IN THE REJECTION
REGION AND [illegible]
FALLS IN THE ACCEPTANCE REGION.
($x_1, \ldots, x_n$ [illegible] POINT IN n-DIMENSIONAL
[illegible] SPACE.)

<u>DEFINITION</u>: A TYPE I ERROR IS THE
ERROR OF REJECTING THE HYPOTHESIS
WHEN [illegible] TYPE II ERROR
IS THE ERROR OF ACCEPTING THE
HYPOTHESIS WHEN IT IS FALSE.

[illegible]
ACCEPT H_0 [illegible]
REJECT H_0 IF $\bar{x} > 64$
TYPE I ERROR IS REJECTING
H_0 IF $\mu \leq 64$
TYPE II ERROR IS ACCEPTING
H_0 IF $\mu > 64$

		STATE OF NATURE	
		H_0 TRUE	H_0 FALSE
DECISION	H_0 TRUE	NO ERROR	TYPE II ERROR
MADE	H_0 FALSE	TYPE I ERROR	NO ERROR

REMARK: WE OFTEN [illegible] THE [illegible] H_0 TO [illegible] [illegible] IN QUE[illegible] [illegible] WILL [illegible] THE [illegible] OF H_0 [illegible] THE SET OF ALTERNATIVES OF INTEREST [illegible] IS [illegible] DENOTED H_1 IS CALLED [illegible] H_0 [illegible] [illegible] SE[illegible] REJECTION (OR CRIT[illegible] REGION) SO THAT

$\alpha = P[\text{TYPE I ERROR}]$

[illegible] EN [illegible] NUMBER.

FROM KNOWING EXAMPLE
EX [illegible] $\alpha = P[\text{TYPE I ERROR}] = 0.05$

[illegible] $= P(\bar{X} > k$ WHEN $\mu = 63) = 0.05$

$$= P\left[Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} > \frac{k-63}{[illegible]}\right] = 0.05$$

$$= P[Z > 2(k-63)]$$

$$Z_{0.05} = 2(k-63) = 1.645$$

$$\Rightarrow k \sim 63.82$$

$\therefore$ REJECT H_0 IF $\bar{X} > 63.82$

5-2-73

P. 256

a) $P[\sigma^2 \le L_2] = 1-\alpha$; $N(\mu, \sigma^2)$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$P[\chi^2_{n-1} \ge \chi^2_{n-1,1-\alpha}] = 1-\alpha$

$= P\left[\frac{(n-1)S^2}{\sigma^2} \ge \chi^2_{n-1,1-\alpha}\right] = 1-\alpha$

$= P\left[\frac{\sigma^2}{S^2(n-1)} \le \frac{1}{\chi^2_{n-1,1-\alpha}}\right] = 1-\alpha$

$\therefore P\left[\sigma^2 \le \frac{(n-1)S^2}{\chi^2_{n-1,1-\alpha}}\right] = 1-\alpha$

b) SAMPLE n TIMES EACH FROM $N(\mu_x, \sigma^2)$ & $N(\mu_y, \sigma^2)$

$T_\gamma = Z\Big/\sqrt{\chi^2_\gamma/\gamma}$; $\frac{(n-1)S_x^2}{\sigma^2} \sim \chi^2_{n-1}$, $\frac{(n-1)S_y^2}{\sigma^2} \sim \chi^2_{n-1}$

$\bar{X} \sim N(\mu_x, \sigma^2/n)$; $\bar{Y} \sim N(\mu_y, \sigma^2/n)$

$\bar{X}-\bar{Y} \sim N(\mu_x-\mu_y, \frac{2\sigma^2}{n})$

$\therefore$ LET $Z = \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\frac{2}{n}}\,\sigma}$

$\frac{(n-1)(S_x^2+S_y^2)}{\sigma^2} \sim \chi^2_{2n-2}$

PUT IT ALL TOGETHER

$T_{2(n-1)} = \frac{Z}{\sqrt{\chi^2_{2n-2}/2n-2}}$

$= \frac{\frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{\frac{2}{n}}}}{\sqrt{\frac{S_x^2+S_y^2}{2}}}$

$= \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{\sqrt{S_x^2+S_y^2/n}}$ (COVER)

(FOR $n_1 \ne n_2$) $\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\frac{(n_1-1)S_x^2+(n_2-1)S_y^2}{n_1+n_2-2}$

$S^2 = \frac{(n_1-1)S_x^2+(n_2-1)S_y^2}{n_1+n_2-2} = \frac{S_x^2+S_y^2}{n}$ FOR $n_1 = n_2$

$\Rightarrow T_{n_1+n_2-2} = \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$

$P\left[\bar{X}-\bar{Y} - \ldots\right]$

$\leq \mu_x - \mu_y \leq \bar{X}-\bar{Y} + t_{\ldots,\alpha/2}\sqrt{\ldots}\ \Big] = 1-\alpha$

Pg. 250

a) $\ldots 10 \Rightarrow \frac{\alpha}{2} = 0.05$

$\bar{X} \pm t \ldots S/\sqrt{n}$

$t_{9,0.05} = 1.833$

$\Rightarrow \bar{X} \pm (1.833)\, S/\sqrt{10}$

$L = \ldots$

$= 3.166 \ldots$

$P(L > 1.17) = P\left[\ldots S > 1.17\right]$

$= P\left[\ldots > 1.17^2\right]$

$= P\left[\frac{121}{90}\, S^2 > 1.17^2\right]$

$= P\left[S^2 > \frac{(1.17)^2\, 90}{121}\right]$

$= P\left[\frac{S^2(n-1)}{\sigma^2} > \frac{(1.17)^2\, 90\,(n-1)}{121\ \sigma^2}\right]$

FOR $\ldots = 1$

$P[L > 1.17] = P\left[\frac{9S^2}{\sigma^2} > \frac{9(1.17)^2\, 90}{121}\right]$

$= P\left[\chi^2_9 > 9.15\right] \ldots$

7) LET R = RELIABILITY

$R_t = P[T > t]$

$R_{10^3} = P[T > 10^3]$

$= \int_{1000}^{\infty} \lambda e^{-\lambda t} \, dt$

$= -e^{-\lambda t} \Big|_{1000}^{\infty} = e^{-1000\lambda}$

FROM THEOREM

$P\left[\lambda < \frac{\chi^2_{2r;1-\alpha}}{2T}\right] = 1 - \alpha$

$R = e^{-1000\lambda} \Rightarrow \ln R = -1000\lambda$

$\Rightarrow \lambda = -\ln R / 10^3$

$P\left[-\frac{\ln R}{10^3} < \frac{\chi^2 \text{[illegible]}}{2T}\right] = 1 - \alpha$

$= P\left[R = e^{-10^3 \lambda} > e^{-10^3 \frac{\chi^2 \text{[illegible]}}{2T}}\right]$

5-3-73

$H_0: \mu = 63 \qquad \sigma = 2$

$H_1: \mu > 63 \qquad n = 16$

REJECT H_0 IF $\bar{X} > 63.82$ GIVES

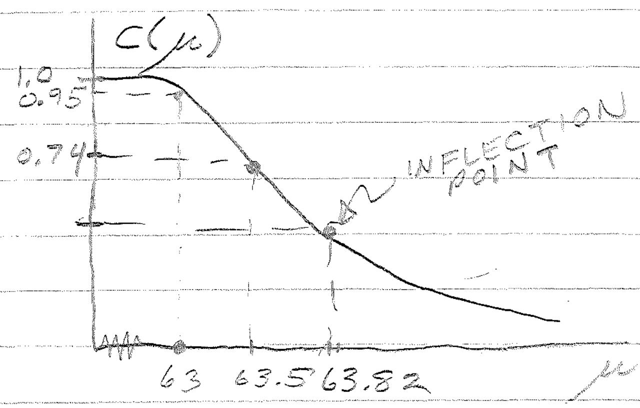
$\alpha = P[\text{TYPE I ERROR}] = 0.05$

WHAT IS $\beta = P[\text{TYPE II ERROR}]$?

DEFINITION: SUPPOSE WE ARE TESTING A HYPOTHESIS H_0 ABOUT A PARAMETER θ THE PROBABILITY OF ACCEPTING H_0 IS A FUNCTION OF θ, DENOTED BY $C(\theta)$, AND IT'S GRAPH IS CALLED THE OPERATING CHARACTERISTIC CURVE FOR THE TEST. $Q(\theta) = 1 - C(\theta)$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \theta]$

IS THE PWR. FUNCTION OF THE TEST.



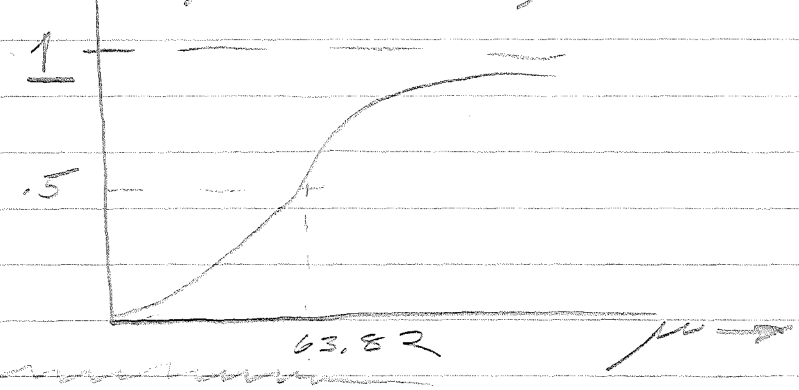
$C(\mu)$	μ
STRAIGHT OFF 63	0.95
63.82	0.5
63.5	0.74
65	

$C(63.5) = P[\text{ACCEPTING } H_0 \text{ WHEN } \mu = 63.5]$

$= P[\bar{X} < 63.82 \text{ WHEN } \mu = 63.5] = P\left[z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{63.82 - 63.50}{2/\sqrt{16}}\right]$

$= P[z < 0.64] = 0.74$

$Q(\mu) = 1 - C(\mu)$ (PWR CURVE



$H_0: \mu = 63$

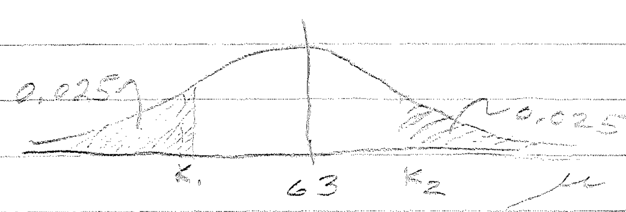
$H_1: \mu \neq 63$ $\therefore$ TWO TAIL TEST

REJECT H_0 IF $\bar{X} < K_1$ OR $\bar{X} > K_2$

WHERE $|63 - K_1| = |63 - K_2|$

LET $\alpha = 0.05$

IF H_0 IS TRUE



$P[\bar{X} > K_2 \text{ GIVEN } \mu = 63] = 0.025$

$= P\left[Z > \frac{K_2 - 63}{2/\sqrt{16}}\right] = 0.025$

$Z_{0.025} = 2$

$\therefore \frac{K_2 - 63}{2/4} = 2 = 64$

AND OBVIOUSLY $K_1 = 62$

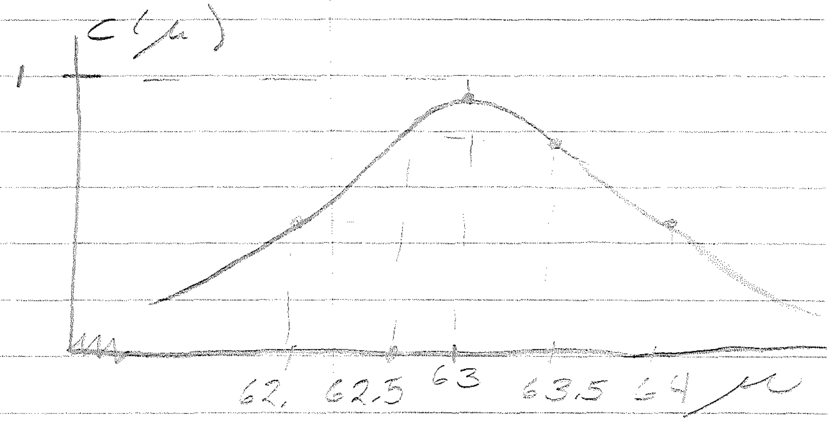
REJECT H_0 IF $\bar{X} > 64$ OR $\bar{X} < 62$

$C(\mu) = P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu]$

$= P[62 < \bar{X} < 64 \text{ GIVEN } \mu]$

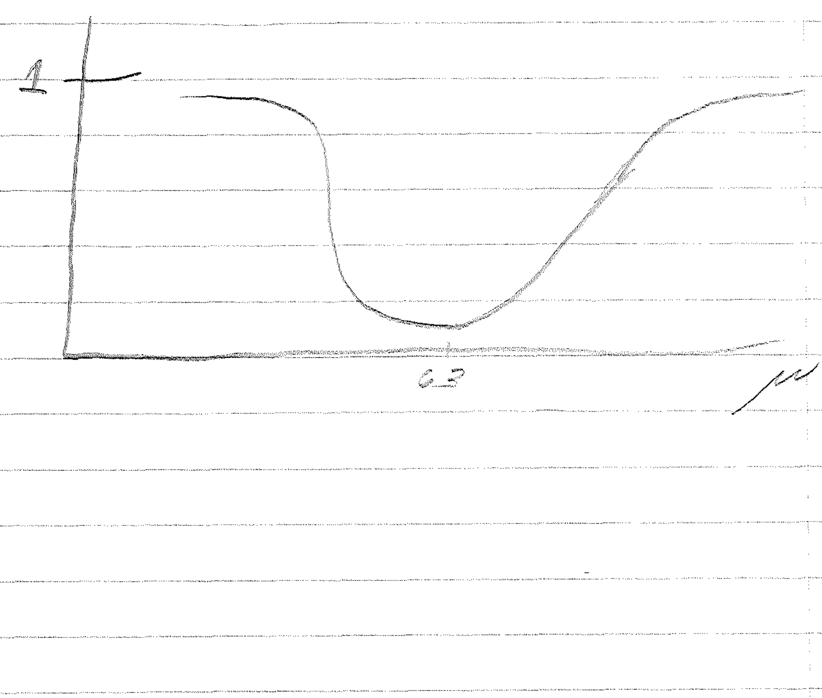
(CONT.)

μ	$C(\mu)$
63	0.95
62.5	0.84
62	0.5
63.5	0.84



$C(62.5) = P[62 < \bar{X} < 64 \text{ GIVEN } \mu = 62.5]$
$= P[-1 < Z < 3]$
$= 0.9987 - 0.1584 = 0.84$

$Q(\mu) = 1 - C(\mu)$ (OWA CURVE)



SEC. 8.1 Pg 258 #1-6

5-4-73

THEOREM (NEYMAN-PEARSON LEMMA) LET $X_1, X_2, \ldots, X_n$ BE A R.S. OF X WHICH HAS ONE UNKNOWN PARAMETER θ, AND LET $L(\theta, x_1, x_2, \ldots, x_n)$ BE THE LIKELIHOOD FUNCTION OF THE SAMPLE. SUPPOSE WE WANNA TEST $H_0: \theta = \theta_0$ AGAINST $H_1: \theta = \theta_1$, WHERE θ_0 & θ_1 ARE GIVEN CONSTANTS. LET S BE THE SAMPLE SPACE FOR THE EXPERIMENT OF TAKING A RANDOM SAMPLE, & LET R (THE REJECTION REGION) BE A SUBSET OF S SUCH THAT

$$\frac{L(\theta_0, x_1, x_2, \ldots, x_n)}{L(\theta_1, x_1, x_2, \ldots, x_n)} \leq K$$

$\ni K$ IS CHOSEN $\ni$ THE PROB. OF TYPE I ERROR IS α. THEN R IS THE MOST POWERFUL CRITICAL (OR REJECTION) REGION AMONG ALL CRITICAL REGIONS OF SIZE α i.e., R IS THE CRITICAL REGION THAT MINIMIZES $\beta = P[\text{TYPE II ERROR}]$

(NO PROOF OFFERED)

THEOREM: ASSUME $X_1, X_2, \ldots, X_n$ IS A R.S. OF $X \sim N(\mu, \sigma^2)$ R.V. THEN THE MOST POWERFUL CRITICAL REGION OF SIZE α FOR TESTING $H_0: \mu = \mu_0$ AGAINST $H_1: \mu = \mu_1 \ni \mu_1 > \mu_0$ IS SPECIFIED BY

$$\bar{X} > \mu_0 + \sigma_0/\sqrt{n}\; Z_\alpha$$

PROOF: SINCE BOTH H_0 & H_1 ARE SIMPLE HYPOTH, WE MAY APPLY THE NEYMANN-PEARSON LEMMA:

$$L(\mu, x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_0} \exp\left[-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma_0}\right)^2\right]$$

$$= \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_0}\right)^n \exp\left[-\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_0)^2\right]$$

$$\therefore \frac{L(\mu_0)}{L(\mu_1)} = \exp\left[-\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_0)^2 + \frac{1}{2\sigma_0^2} \sum_{i=1}^{n} (x_i - \mu_1)^2\right]$$

$$= \exp\left[-\frac{1}{2\sigma_0^2}\left(2\sum x_i(\mu_1 - \mu_0) + n(\mu_0^2 - \mu_1^2)\right)\right]$$

$$= \exp\left[\frac{-1}{2\sigma_0^2}\left\{2\sum x_i(\mu_1 - \mu_0)\right\}\right] \exp\left[\frac{-1}{2\sigma_0^2}\left\{n(\mu_0^2 - \mu_1^2)\right\}\right]$$

REJECT H_0 IF $\exp\left[\frac{-\sum x_i}{\sigma_0^2}(\mu_1 - \mu_0)\right] < k'$

WHERE $k' = k\, e^{\frac{n}{2\sigma_0^2}(\mu_0^2 - \mu_1^2)}$

$\Rightarrow$ REJECT H_0 IF $\frac{1}{\sigma_0^2}(\mu_1 - \mu_0)\sum x_i \geq k''$; $k'' = \ln k'$

IF $\sum x_i \geq k''' \ni k''' = \frac{k'' \sigma_0^2}{\mu_1 - \mu_0}$

IF $\bar{x} > k^{iv} \ni k^{iv} = k'''/n$

$\therefore \bar{X}$ DEFINES THE MOST PWRFUL CRITICAL REGION

$$P[\bar{X} > k \mid \overset{\text{GIVEN}}{\mu = \mu_0}] = \alpha \Rightarrow k = Z_\alpha \frac{\sigma_0}{\sqrt{n}} + \mu_0$$

THEOREM: ASSUME $X_1, X_2, \ldots, X_n$ IS A R.S. OF $X \sim N(\mu, \sigma_0^2)$ R.V. THEN THE MOST PWRFUL CRITICAL REGION OF SIZE α FOR TESTING $H_0: \mu = \mu_0$ VS $H_1: \mu = \mu_1 \ni \mu_1 < \mu_0$ IS SPECIFIED BY $\bar{X} < \mu_0 - \frac{\sigma_0}{\sqrt{n}} Z_\alpha$

PROOF: SINCE BOTH H_0 & H_1 ARE SIMPLE, WE MAY APPLY THE NEYMANN PEARSON LEMMA & BY A SIMILAR APPROACH AS BEFORE EXCEPT THE QUANTITY $(\mu_1 - \mu_0) < 0$. $\therefore$ WE OBTAIN $\bar{X} < k$, SO H_0 IS REJECTED IF $\bar{X} < k$.

$$P[\bar{X} < k \text{ GIVEN } \mu = \mu_0] = \alpha \Rightarrow k = \mu_0 - Z_\alpha \sigma_0/\sqrt{n}$$

NOTE: THE NEYMAN PEARSON LEMMA IS VALID FOR:

$H_0: \theta = \theta_0$	$H_0: \theta = \theta_0$	$H_0: \theta = \theta_0$	$H_0: \theta \leq \theta_0$
$H_1: \theta = \theta_1$	$H_1: \theta > \theta_0$	$H_1: \theta < \theta_0$	$H_1: \theta > \theta_0$

THESE ALL HAVE A UNIFORMLY MOST PWRFUL TEST

$H_0: \theta = \theta_0$; $H_1: \theta \neq \theta_0$ DOES NOT

<u>THEOREM</u>: ASSUME $X_1, X_2, \ldots, X_n$ IS A R.S. OF $X \sim N(\mu, \sigma^2)$ $\ni \mu$ IS KNOWN, THE MOST PWRFUL CRITICAL REGION OF SIZE α FOR TESTING $H_0: \sigma = \sigma_0$ VS. $H_1: \sigma = \sigma_1 \ni \sigma_1 > \sigma_0$ IS SPECIFIED BY $\sum_{i=1}^{n} (X_i - \mu)^2 > \sigma_0^2 \chi^2_\alpha$

<u>PROOF</u>: BOTH H_0 & H_1 ARE SIMPLE, APPLY NEYMAN-PEARSON LEMMA

$L(\sigma) = \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\sigma}\right]^n \exp\left[-\frac{1}{2\sigma^2} \sum (X_i - \mu)^2\right]$

$\therefore L(\sigma_0)/L(\sigma_1) = (\sigma_1/\sigma_0) \exp\left[-\frac{1}{2} \sum (X_i - \mu)^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right]$

REJECT H_0 IF $\frac{L(\sigma_0)}{L(\sigma_1)} \leq K$

OR $\exp\left[-\frac{1}{2} \sum (x_i - \mu)^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right] \leq K'$

OR $\frac{1}{2} \sum (X_i - \mu)^2 \left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > K''$

OR $\sum (X_i - \mu)^2 > K'''$

$P[\text{TYPE I ERROR}] = P\left[\sum (X_i - \mu)^2 > K''' \text{ GIVEN } \sigma = \sigma_0\right] = \alpha$

$= P\left[\frac{\sum (X_i - \mu)^2}{\sigma_0^2} > \frac{K'''}{\sigma_0^2}\right] = P\left[\chi^2_n > \frac{K'''}{\sigma_0^2}\right] = \alpha$

$\Rightarrow K''' = \sigma_0^2 \chi^2_{n;\alpha}$

$\Rightarrow$ REJECT H_0 IF $\sum (X_i - \mu)^2 > \sigma_0^2 \chi^2_{n,\alpha}$

<u>THEOREM</u>: ASSUME $X_1, X_2, \ldots, X_n$ IS A R.S. OF $X \sim N(\mu, \sigma^2)$ R.V., $\ni \mu$ IS KNOWN, THEN THE MOST PWRFUL CRITICAL REGION OF SIZE α FOR TESTING $H_0: \sigma = \sigma_0$ VS $H_1: \sigma = \sigma_1 \ni (\sigma_1 < \sigma_0)$ IS GIVEN BY $\sum (X_i - \mu)^2 < \sigma_0^2 \chi^2_{1-\alpha}$

<u>PROOF</u>: FOLLOWING THE SAME PROCEDURE AS ABOVE BUT NOTING THAT $\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} < 0\right)$, WE HAVE:

REJECT H_0 IF $\sum (X_i - \mu)^2 < K'''$

$\Rightarrow P[\text{TYPE I ERROR}] = \alpha$

$= P\left[\sum (X_i - \mu)^2 < K''' \text{ GIVEN } \sigma = \sigma_0\right]$

$= P\left[\frac{\sum (X_i - \mu)^2}{\sigma_0^2} < \frac{K'''}{\sigma_0^2}\right] = \alpha = P\left[\chi^2_n < \frac{K'''}{\sigma_0^2}\right]$

$\therefore K''' = \sigma_0^2 \chi^2_{n;1-\alpha}$

$\Rightarrow$ REJECT H_0 IF $\sum (X_i - \mu)^2 < \sigma_0^2 \chi^2_{n;1-\alpha}$

5-8-73

$H_0: \mu = \mu_0$

$H_1: \mu = \mu_1 \ni \mu_1 > \mu_0$

σ_0 IS KNOWN; SAMPLE n FROM A $N(\mu, \sigma_0^2)$

REJECT H_0 IF $\dfrac{L(\mu_0, x_1, x_2, \ldots, x_n)}{L(\mu_1, x_1, x_2, \ldots, x_n)} \leq k$

OR REJECT H_0 IF $\bar{x} > K = ?$

$P[\bar{x} > K \text{ GIVEN } \mu = \mu_0] = \alpha$

$= P\left[z > \dfrac{K - \mu_0}{\sigma_0/\sqrt{n}}\right] = \alpha$

$= P[z > z_\alpha] = \alpha$

$\therefore \dfrac{K - \mu_0}{\sigma_0/\sqrt{n}} = z_\alpha$

$\Rightarrow K = \mu_0 + z_\alpha \sigma_0/\sqrt{n}$; (MOST POWERFUL TEST)

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$ $\forall \mu_1 > \mu_0$, WE GET MAXIMUM PWR BY REJECTING $\bar{x} > K$

$H_0: \mu = \mu_0$

$H_1: \mu = \mu_1 \; (\mu_1 < \mu_0)$

REJECT H_0 IF $\bar{x} < K$

COMPARE WITH

$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$

TROUBLE:

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$ A MOST PWRFUL TEST DOESN'T XIST

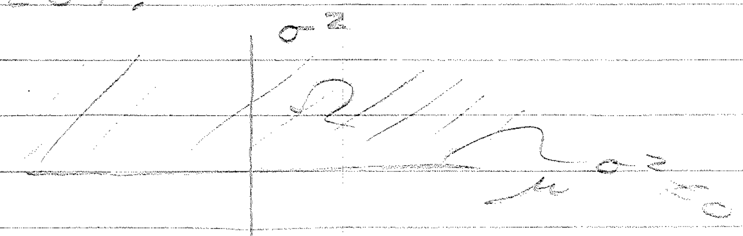
DEFINITION: LET $X_1, X_2, \ldots, X_n$ BE A RANDOM SAMPLE OF X, WHOSE DENSITY OR PROBABILITY FUNCTION DEPENDS ON THE PARAMETER $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$

LET Ω BE THE UNIVERSAL SET OF POSSIBLE VALUES OF $\underline{\theta}$ WHICH ARE OF INTEREST.

EX) $N(\mu, \sigma^2)$

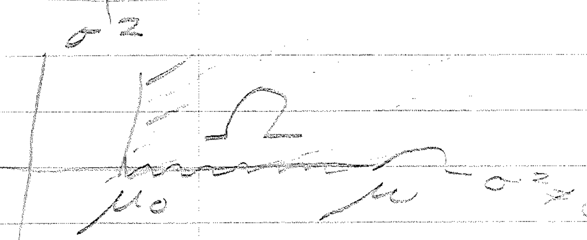
a) $H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$



b) $H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$



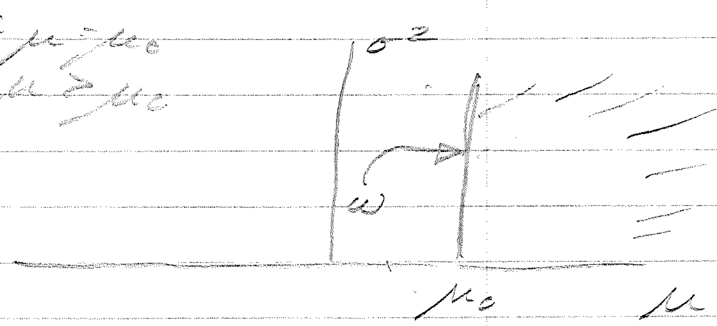
SUPPOSE WE WANNA TEST

$H_0: \underline{\theta} \in \omega$ AGAINST $H_1: \underline{\theta} \in \omega'$

$[\omega \cup \omega' = \Omega$ & $\omega \cap \omega' = \phi]; \omega \in \Omega$

EX) $H_0: \mu = \mu_0$

$H_1: \mu \geq \mu_0$



LET $L(\underline{\theta})$ BE THE LIKELIHOOD FUNCTION AND LET $L(\hat{\omega})$ BE THE MAXIMUM OF $L(\theta_1)$ WITHIN ω,

(CONT.)

AND $L(\hat{\Omega})$ BE THE MAXIMUM $L(\theta)$ WITHIN Ω. THEN THE LIKELIHOOD RATIO TEST H_0 AGAINST H_1 IS TO REJECT H_0 WHEN $\lambda = L(\hat{\omega})/L(\hat{\Omega}) \leq A \ni$ A IS CHOSEN SO THAT THE PROBABILITY OF TYPE I ERROR IS α.

ASSIGN. Pp. 267-8 PROB 1-8
Pp. 278-9 PROB 1-8

5-9-73 LIKELIHOOD RATIO TEST (GENERAL)

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \qquad H_0: \theta \in \omega$$
$$H_1: \theta \in \omega'$$

$\omega \cup \omega' = \Omega$; $\omega \cap \omega' = \emptyset$

REJECT H_0 IF $\lambda < A \ni A$ IS A CONSTANT DETERMINED BY α

N.-PIERSON LEMMA (MORE SPECIFIC, BUT MORE POWERFUL)

$H_0: \theta = \theta_0$
$H_1: \theta = \theta_1$

$$\Rightarrow \lambda = \frac{L(\theta_0)}{\max\limits_{\theta_0, \theta_1} L(\theta)}$$

NOTE

$0 \leq \lambda \leq 1$ (ALWAYS)

$0 < \lambda < 1$ (CONTINUOUS POP.)

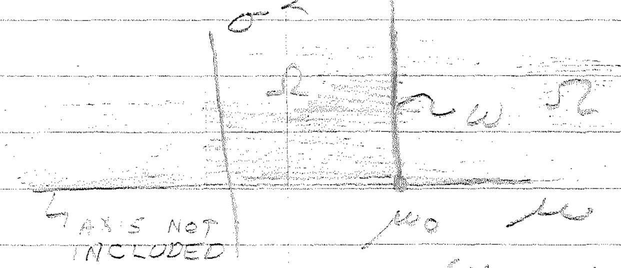
SO THAT FROM CONT. POP.:

$$\lambda = \frac{L(\theta_0)}{L(\theta_1)} \quad \text{SINCE}$$

$\max\limits_{\theta_0, \theta_1} L(\theta) = L(\theta_0)$ WILL NEVER OCCUR

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

$N(\mu, \sigma^2)$: BOTH μ & σ^2 UNKNOWN



$$L(\mu, V) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \frac{1}{V^{1/2}} e^{-\frac{1}{2}\frac{(x_i-\mu)^2}{V}}$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{V^{n/2}} e^{-\frac{1}{2}\frac{\Sigma(x_i-\mu)^2}{V}}$$

$$L(\hat{\Omega}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\hat{V}^{n/2}} e^{-\frac{1}{2}\frac{\Sigma(x_i-\hat{\mu})^2}{\hat{V}}}$$

$$\Rightarrow \hat{\mu} = \bar{x}, \quad \hat{V} = \frac{\Sigma(x_i-\bar{x})^2}{n}$$

$$\Rightarrow L(\hat{\Omega}) = \frac{1}{(2\pi)^{n/2}} \left[\frac{n}{\Sigma(x_i-\bar{x})^2}\right]^{n/2} e^{-\frac{1}{2}n}$$

IN ω, $L(V) = \frac{1}{(2\pi)^{n/2}} \frac{1}{V^{n/2}} e^{-\frac{1}{2}\frac{\Sigma(x_i-\mu_0)^2}{V}}$

IN ω, $\hat{V} = \frac{\Sigma(x_i-\mu_0)^2}{n}$

$$L(\hat{\omega}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\hat{V}^{n/2}} e^{-\frac{1}{2}\frac{\Sigma(x_i-\mu_0)^2}{\hat{V}}}$$

$$= \frac{1}{(2\pi)^{n/2}} \left[\frac{n}{\Sigma(x_i-\mu_0)^2}\right]^{n/2} e^{-\frac{1}{2}n}$$

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left[\frac{\Sigma(x_i-\bar{x})^2}{\Sigma(x_i-\mu_0)^2}\right]^{n/2}$$

(CONT)

AGAIN
$$\lambda = \left[\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \mu_0)^2}\right]^{n/2}$$

$$\sum (x_i - \mu_0)^2 = \sum \left[(x_i - \bar{x}) + (\bar{x} - \mu_0)\right]^2$$

$$= \sum (x_i - \bar{x})^2 + \sum (\bar{x} - \mu_0) + 2 \sum (\bar{x} - \mu_0)(x_i - \bar{x})$$
$$= \sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0) + 2(\bar{x} - \mu_0) \sum (x_i - \bar{x})$$
$$= \sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)$$

$$\Rightarrow \lambda = \left[\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}\right]^{n/2}$$
$$= \left[\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}}\right]^{n/2}$$

NOTE
$$t_{n-1} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \Rightarrow (t_{n-1})^2 = \frac{n(\bar{x} - \mu_0)^2}{s^2}$$

$$= \frac{n(\bar{x} - \mu_0)}{\frac{\sum (x_i - \bar{x})^2}{n-1}}$$

AND $$\lambda = \left[\frac{1}{1 + \frac{t^2}{(n-1)}}\right]^{n/2}$$ (IF H_0 IS TRUE)

REJECT H_0 WHEN $\lambda < A$

$$\left[\frac{1}{1 + \frac{t^2}{n-1}}\right]^{n/2} < A$$

$$\Rightarrow t^2 > \frac{(n-1)(1 - A)^{2/n}}{A^{2/n}} = c$$
$$c' = \sqrt{c}$$

REJECT H_0 WHEN $t > c'$ OR $t < c'$

RECALL $H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$
COMPUTE $t = \bar{x} - \mu_0 / s/\sqrt{n}$

CHOOSE $c' \ni P[\text{TYPE I ERROR}] = \alpha$
$\Rightarrow$ REJECT H_0 IF $t > t_{n-1;\alpha/2}$ OR
IF $t < -t_{n-1;\alpha/2}$

5-15-73

FINAL ~ [illegible] OPEN BOOK

COMPUTATIONAL TOPICS

1) CENTRAL LIMIT THEOREM

a) N(0,1) REPRESENTING BINOMIAL ($\frac{1}{2}$ CORRECTION FACTOR)

$P_{\bar{s}} = 3$

i) $n = 180$; $p = 1/6$

$P[X_b \geq 25]$

$\approx P[X \geq 24.5$

$\mu = np = 30$; $\sigma^2 = np(1-p) = 25$

$\sigma = 5$

$= P[\bar{X}_n \geq 24.5]$

$= P[Z \geq \frac{24.5 - 30}{5}]$

b) POISSON-'N' APPROXIMATION

2) $\bar{X} \sim N(\mu, \sigma^2/n)$

3) CONF. INTERVAL

P_5 97

1) BERNOULLI R.V.

$$P_x(x) = \begin{cases} P & ; x = 1 \\ 1-P & ; x = 0 \end{cases}$$

$P_x(x)$	x
$1/6$	1
$5/6$	0

$$M_x(t) = E[e^{xt}]$$

$$= \sum_{x=0}^{1} e^{xt} P_x(x)$$

$$= \left[1 \times \frac{5}{6}\right] + \left[e^t \times \frac{1}{6}\right]$$

$$= \frac{5}{6} + \frac{1}{6} e^t$$

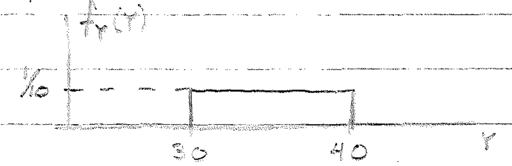
$$m_1 = \frac{\partial M_x(t)}{dt}\bigg|_{t=0} = \frac{1}{6} e^{(0)} = \frac{1}{6}$$

$$m_2 = \frac{1}{6}$$

$$m_3 = \frac{1}{6}$$

Pg 98

2)



$$M_Y(t) = E\left[e^{Yt}\right]$$

$$= \int_{30}^{40} \frac{e^{yt}}{10}\, dy$$

$$= \frac{t\, e^{yt}}{10}\Bigg|_{30}^{40}$$

$$= \frac{t}{10}\left[e^{40t} - e^{30t}\right]$$

$$M_{Y-\mu}(t) = e^{-\mu t} M_Y(t)$$

$$\mu = 35$$

$$\Rightarrow M_{Y-\mu}(t) = e^{-35t}\left[\frac{t}{10}\left(e^{40t} - e^{30t}\right)\right]$$

$$= \frac{t}{10}\left[e^{5t} - e^{-5t}\right]$$

$$= \frac{t}{5} \sinh(5t)$$

Pg 98

3) $f_Z(z) = 500\, e^{-500z}$

$$M_x(t) = E[e^{zt}]$$
$$= \int_{-\infty}^{\infty} f_Z(z)\, e^{zt}\, dz$$
$$= \int_{0}^{\infty} 500\, e^{-500z}\, e^{zt}\, dz$$
$$= 500 \int_{0}^{\infty} e^{(t-500)z}\, dz$$
$$= -\frac{500}{t-500} \quad ; \quad t < 500$$

$$m_1 = \mu = \left.\frac{\delta M_x(t)}{\delta t}\right|_0 = +\left.\frac{500}{(t-500)^2}\right|_{t=0} = \frac{1}{500}$$

$$m_2 = E[x^2] = \frac{\delta^2 M_x(t)}{\delta t} = -\frac{500}{(t-500)^3}\, \frac{\delta}{dt}(t-500)^2 \Big|_{t=0}$$
$$= -\frac{500}{(t-500)^3}\,[2t-1000]\,\Big|_{t=0}$$
$$= \frac{500 \times 2}{(500)^3} = \frac{2}{500^2}$$

$$\sigma^2 = m_2 - m_1^2 = 1/(500)^2$$

Pg 112

10) $M_x(t) = (pe^t + q)^n$ $\ni q = 1 - p$

$\mu = np$; $\sigma = \sqrt{npq}$

$$M_{\frac{x-\mu}{\sigma}}(t) = e^{-\frac{\mu}{\sigma}t} M_x\left(\frac{t}{\sigma}\right)$$

$$= e^{-\frac{np}{\sqrt{nqp}}t}\left(pe^{t/\sqrt{npq}} + q\right)^n$$

$$= e^{-\sqrt{\frac{np}{q}}t}\left(pe^{\frac{t}{\sqrt{npq}}} + q\right)^n$$

$$= \left[e^{-\frac{1}{n}\sqrt{\frac{np}{q}}t}\left(pe^{\frac{t}{\sqrt{npq}}} + q\right)\right]^n$$

$$= \left[e^{-\sqrt{\frac{p}{nq}}t}\left(pe^{\frac{t}{\sqrt{npq}}} + q\right)\right]^n$$

Pg 129

7) $f_x(x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$$M_x(t) = E[e^{xt}]$$

$$= \sum_{x=0}^{\infty} e^{xt} f_x(x)$$

$$= \sum e^{xt}\left[\frac{e^{-\lambda}\lambda^x}{x!}\right]$$

$$= \sum e^{-\lambda}(\lambda e^t)^x / x!$$

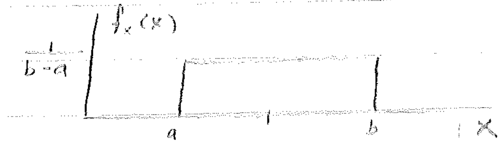
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

Pg 136

3)



a) $M_X(t) = \int_a^b \left(\frac{1}{b-a}\right) e^{xt}\, dX$

$= \frac{1}{(b-a)} \frac{1}{t} e^{Xt}\Big|_a^b$

$= \frac{1}{b-a} \frac{1}{t} \left[e^{bt} - e^{at}\right]$

b) $\frac{dM_X(t)}{dt} = \frac{1}{(b-a)} \frac{t[b e^{bt} - a e^{at}] - [e^{bt} - e^{at}]}{t^2}$

$= \frac{1}{(b-a)} \frac{[bt-1]e^{bt} + [-at+1]e^{at}}{t^2}$

$\lim_{t\to 0} \frac{dM_X(t)}{dt} = \frac{1}{(b-a)} \frac{b e^{bt} + b(bt-1)e^{bt} - a e^{at} + a[-at+1]e^{at}}{2t}$

$= \lim_{t\to 0} \frac{1}{b-a} \frac{b(1+bt-1)e^{bt} + a(-1-at+1)e^{at}}{2t}$

$= \lim_{t\to 0} \frac{1}{2(b-a)}\left[b^2 e^{bt} - a^2 e^{at}\right] = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$

c) $\frac{d^2M_X(t)}{dt} = \left(\frac{1}{b-a}\right) \frac{t^2[b^2 t e^{bt} - a^2 t e^{at}] + 2t[(bt-1)e^{bt} + (-at+1)e^{at}]}{t^4}$

$= \frac{1}{b-a} \frac{\{b^2 \cancel{t^3}^{\,2} + 2\cancel{t}(bt-1)\}e^{bt} + \{a^2 \cancel{t^3}^{\,2} + 2\cancel{t}(-at+1)\}e^{at}}{\cancel{t^4}^{\,3}}$

$= \frac{1}{b-a} \frac{(b^2t^2+2bt-2)e^{bt} + (a^2t^2-2at+2)e^{at}}{t^3}$

$\lim_{t\to 0} \frac{d^2M_X(t)}{dt} = \frac{1}{b-a} (2b^2t+2b)e^{bt} + (b^2t^2+2bt-2)e^{bt}$

FIZZLE

Pg 136

(CONT.)

3) $M_x(t) = \frac{1}{t(b-a)}\left[e^{bt} - e^{at}\right]$

$$= \frac{1}{t(b-a)}\Big[\Big(1 + \frac{bt}{1!} + \frac{(b^2t^2)}{2!} + \frac{(b^3t^3)}{3!} + \dots\Big) - \Big(1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \frac{b^3t^3}{3!} + \dots\Big)\Big]$$

$$= \frac{1}{t(b-a)}\Big[\Big(\frac{bt}{1!} + \frac{(b^2t^2)}{2!} + \frac{(b^3t^3)}{3!} + \dots\Big) - \Big(\frac{at}{1!} + \frac{a^2t^2}{2!} + \frac{b^3t^3}{3!} + \dots\Big)\Big]$$

$$= \frac{1}{t(b-a)}\Big[\frac{(b-a)t}{1!} + \frac{(b^2-a^2)t^2}{2!} + \frac{(b^3-a^3)t^3}{3!} + \dots\Big]$$

$$= \frac{1}{b-a}\Big[(b-a) + \frac{(b^2-a^2)t}{2!} + \frac{(b^3-a^3)t^2}{3!} + \dots\Big]$$

$$\frac{dM_x(t)}{dt}\Big|_{t=0} = \frac{1}{(b-a)}\frac{(b^2-a^2)}{2} = \frac{b+a}{2} = \mu$$

$$\frac{d^2M_x(t)}{dt}\Big|_{t=0} = \frac{1}{(b-a)}\frac{2(b^3-a^3)}{6} = \frac{1}{3}(b+a)^2$$

$$\sigma^2 = m_2 - m_1^2$$

$$= \frac{1}{3}(b+a)^2 - \frac{1}{4}(b+a^2)$$

$$= \frac{1}{12}(b+a)^2$$

Pg 179-80

1) EACH DAY IS A BERNOULLI R.V. WITH $P = 1/2$

$\Rightarrow Y = \sum_{i=1}^{20} X_i$ IS A BINOMIAL R.V.

$$Y = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{20}{x}\left(\tfrac{1}{2}\right)^{20}$$

$$P[Y \geq 10] = -P[Y \leq 9] + 1$$

$$1 - 0.4119 \qquad \text{(TABLE 5A)}$$

$$= 0.0207$$

2) a) $P_{X_i}(x_i) = \dfrac{e^{-\lambda_i}\lambda_i^{x_i}}{x_i!}, \quad x = 0, 1, 2, \ldots$

$$\Rightarrow P_X(Y) = \prod_{i=1}^{3} P_{X_i}(x_i)$$

$$= \frac{e^{-(\lambda_1+\lambda_2+\lambda_3)}\lambda_1^{x_1}\lambda_2^{x_2}\lambda_3^{x_3}}{x_1!\,x_2!\,x_3!}$$

b) $P_Y(Y) = \dfrac{e^{-5(\lambda_1+\lambda_2+\lambda_3)}\,5^{x_1+x_2+x_3}\,\lambda_1^{x_1}\lambda_2^{x_2}\lambda_3^{x_3}}{x_1!\,x_2!\,x_3!}$

3) $\sigma_x = 0.1 \Rightarrow \sigma_x^2 = 0.01$

$\mu_x = 1.05$

$\mu_Y = 5\mu_x = 5.25$

$\sigma_Y^2 = 5\sigma_x^2 = 0.05 \Rightarrow \sigma_Y = 0.224$

$$P[Y > 5] = P\left[Z > \frac{5 - 5.25}{0.224}\right]$$

$$= 0.8686$$

4) $Y = \sum_{i=1}^{n} X_i$

$$M_Y(t) = E[e^{Yt}]$$
$$= E[e^{\sum X_i t}]$$
$$= \prod_{i=1}^{n} E[e^{X_i t}]$$
$$= \prod_{i=1}^{n} M_{X_i}(t)$$

6) $Y = \sum_{i=1}^{k} X_i$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$$

$$M_{X_i}(t) = (pe^t + q)^{n_i}$$
$$\Rightarrow M_Y(t) = \prod_{i=1}^{k} (pe^t + q)^{n_i}$$
$$= (pe^t + q)^{\sum_{i=1}^{k} n_i} \Rightarrow \text{B.R.V. M.G.F. FOR}$$
$$p_Y = p \quad \text{AND} \quad n_Y = \sum_{i=1}^{k} n_i$$

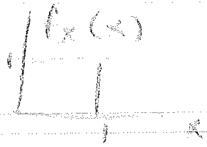
7) $Y = \sum_{i=1}^{n} X_i$

$$M_{X_i}(t) = e^{\lambda_i(e^t - 1)}$$
$$\Rightarrow M_Y(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t - 1)}$$
$$= e^{\sum_{i=1}^{n} \lambda_i (e^t - 1)}$$

$$\Rightarrow \lambda_Y = \sum_{i=1}^{n} \lambda_i$$

Pg 185-6

1)



$P[|X-\mu_x| < k\sigma_x] \geq 1 - 1/k^2$

$\mu = 1/2 \quad ; \sigma^2 = 1/12 \Rightarrow \sigma = \frac{1}{2\sqrt{3}}$

$k = 5/4$

$P[|X - \frac{1}{2}| < \frac{5}{4} \frac{1}{2\sqrt{3}}] \geq 1 - \frac{1}{25/16} = 9/25 = 0.36$

$P[|X - \frac{1}{2}| < 5/4 \frac{1}{2\sqrt{3}}] = 0.722$

$k = 3/2$

$P[|X - \frac{1}{2}| < \frac{3}{4\sqrt{3}}] \geq 1 - 4/9 = 5/9$

$P[|X - \frac{1}{2}| < \frac{3}{4\sqrt{3}}] = 6/4\sqrt{3}$

ETC.

2) $P_X(x) = \frac{k^2-1}{k^2}\delta(x) + \frac{1}{2k^2}\delta(x-k) + \frac{1}{2k^2}\delta(x+k)$

$\mu = \int_{-\infty}^{\infty} x\, P_X(x)\, dx = 0$

$E[x^2] = \int_{-\infty}^{\infty} x^2 P_X(x)\, dx$

$= \frac{2}{2k^2}(k^2) = \underline{1} \Rightarrow \sigma_x = \sigma_x^2 = 1$

$P[|X-\mu_x| < k\sigma_x] \geq 1 - \frac{1}{k^2}$

$P[|X| < k] \geq 1 - \frac{1}{k^2}$

$P[|X| < k] = P[X=0] + P[X=-k]$

$= \frac{k^2-1}{k^2} + \frac{1}{2k^2}$

$= \frac{2k^2-2+1}{2k^2} = 1 - \frac{1}{2k^2}$ O.K.

Pg. 217

1) X_i IS A POISSON R.V. WITH $\mu = \lambda = 1$ AND $\sigma^2 = \lambda = 12$

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ IS A POISSON R.V. WITH $n = 12$,

$$\mu_{\bar{x}} = n\,\mu_i = 12\lambda = 12$$

$$\sigma_{\bar{x}}^2 = n\,\sigma_i^2 = 144 \Rightarrow \sigma_{\bar{x}} = 12$$

$$P[\overline{X} > 1.1] = P[\mu_{\bar{x}} = 12\overline{X} > 13.2]$$
$$= P\left[\sum_{i=1}^{n} X_i > 13.2\right]$$
$$= P\left[\sum_{i=1}^{n} X_i \geq 14\right]$$
$$= 1 - P\left[\sum_{i=1}^{n} \overline{X} < 14\right]$$
$$= 1 - P\left[\sum_{i=1}^{n} \overline{X} \leq 13\right]$$
$$= 1 - 0.682 \qquad (\lambda = 12)$$
$$= 0.318$$

$$P[\overline{X} < 0.85] = P\left[\frac{1}{n}\sum_{i=1}^{n} X_i < 0.85\right]$$
$$= P\left[\sum_{i=1}^{12} X_i < 10.2\right]$$
$$= P\left[\sum_{i=1}^{12} X_i \leq 10\right]$$
$$= 0.347$$

2) X_i IS A $N(\mu, \sigma^2) = N(a, 0.01)$ R.V.

$\Rightarrow \bar{X}$ IS A $N(a, \frac{0.01}{5})$ R.V.

$$\Rightarrow P[a - 0.05 < \bar{X} < a + 0.05]$$

$$= P[-0.05 < \bar{X} - a < 0.05]$$

$$= P\left[\frac{-0.05}{0.1}\sqrt{5} < Z < \frac{0.05}{0.1}\sqrt{5}\right]$$

$$= P[-1.115 < Z < 1.115]$$

$$= P[Z < 1.115] - P[Z < -1.115]$$

$$= P[Z < 1.115] - [1 - P(Z < 1.115)]$$

$$= 2P[Z < 1.115] - 1$$

$$= 2(0.867) - 1$$

$$= 0.734$$

Pg. 217

3) X_i IS A BERNOULLI RV (p)

ΣX_i IS A BINOMIAL RV $\mu = np$; $\sigma^2 = \mu(1-p)$

$\Rightarrow P[\frac{1}{2} - 0.05 \leq \bar{X} \leq \frac{1}{2} + 0.05]$

$= P[0.45 \leq \bar{X} \leq 0.55]$

$= P[2.25 \leq \sum_{i=1}^{5} X_i \leq 2.750] = 0$

$P[\frac{1}{2} - 0.15 \leq \bar{X} \leq \frac{1}{2} + 0.15]$

$= P[0.35 \leq \bar{X} \leq 0.65]$

$= P[1.75 \leq 5\bar{X} = \sum_{i=1}^{5} X_i \leq 3.25]$

$= P[5\bar{X} = 2] + P[5\bar{X} = 3]$

$= P[5\bar{X} \leq 3] - P[5\bar{X} \leq 1]$

$= 0.8125 - 0.1875$

$= 0.6250$

4) FIND $P[a\sigma^2 < S^2 < b\sigma^2] = 0.95$

$Y = \frac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} R.V.

$n = 5 \Rightarrow Y$ IS A χ^2_4 R.V.

$P[a\sigma^2 < S^2 < b\sigma^2]$

$= P[a < \frac{S^2}{\sigma^2} < b]$

$= P[4a < \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} < 4b] = 0.95$

$= P[\chi^2_{4;0.025} < Y < \chi^2_{4;.975}]$

$a = \frac{0.484}{4} = 0.121$; $b = \frac{11.12}{4} = 2.775$

5) $F_{X_{(n)}}(t) = \left[F_X(t)\right]^n$

$$F_X(t) = \begin{cases} t \; ; & 0 < t < 1 \\ 0 \; ; & \text{OTHERWISE} \end{cases}$$

$$\Rightarrow F_{X_{(n)}}(t) = \begin{cases} t^n = t^{10} \; ; & 0 < t < 1 \\ 0 \; ; & \text{OTHERWISE} \end{cases}$$

$$f_{X_{(n)}}(t) = \frac{d}{dt} F_{X_n}(t)$$

$$= \begin{cases} 10t^9 \; ; & 0 < t < 1 \\ 0 \; ; & \text{OTHERWISE} \end{cases}$$

$$P[X_{(n)} > 0.9] = \int_{0.9}^{1} 10t^9 \, dt$$

$$= t^{10} \Big|_{0.9}^{1.0}$$

$$= 1 - (0.9)^{10}$$

$$P[X_{(n)} < 0.5] = \int_{0}^{0.5} 10t^9 \, dt$$

$$= t^{10} \Big|_{0}^{0.5}$$

$$= (0.5)^{10}$$

6) $F_{X_{(1)}}(t) = 1 - [1 - F_X(t)]^n$

$$= \begin{cases} 1-(1-t)^{10} & ; \quad 0 \le t < 1 \\ 0 & ; \quad \text{OTHERWISE} \end{cases}$$

$$f_{X_{(1)}}(t) = \frac{d}{dt} F_{X_{(1)}}(t)$$

$$= 10(1-t)^9$$

$$P[X_{(1)} < 0.5] = \int_0^{0.5} 10(1-t)^9 \, dt$$

$$= -(1-t)^{10} \Big|_0^{0.5}$$

$$= [-(1-0.5)^{10}] - [-1^{10}]$$

$$= 1-(0.5)^{10}$$

$$P[X_{(n)} > 0.9] = \int_{0.9}^{1} 10(1-t)^9 \, dt$$

$$= -(1-t)^{10} \Big|_{0.9}^{1}$$

$$= [1-0.9]^{10}$$

$$= (0.1)^{10}$$

$$= 10^{-10}$$

7) WE HAVE 32 INDEPENDENT BERNOULLI R.V. WITH $p = 0.3$.

FIND $P[\bar{X} < 1.5]$ AND $P[\bar{X} > 2]$

$Y = \sum_{i=1}^{32} X_i$ IS A BINOMIAL R.V. WITH

$n = 32$ & $p = 0.3$

$\mu = np \qquad \sigma^2 = np(1-p)$

$$P[\bar{X} < a] \approx P\left[Z < \frac{a-\mu}{\sigma}\right]$$

ETC.

8) $P(X_{(1)} = 1) = P[X_{(1)} = 1 \text{ AND } X_{(2)} = 1, \ldots, X_{(n)} = 1]$

$= p^n$

$P[X_{(n)} = 1] = 1 - P[X_{(n)} = 0]$

$P[X_{(n)} = 0] = (1 - p)^n$

$\Rightarrow P[X_{(n)} = 1] = 1 - (1 - p)^n$

9) $$F_{X_i}(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$f_{X_i}(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{dF_{X_i}(t)}{dt}$$

$$F_{X_{(n)}}(t) = \left[F_X(t)\right]^n$$

$$= \left[\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right]^n$$

$$f_{X_n}(t) = \frac{d}{dt} F_{X_{(n)}}(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right]^n$$

$$= n\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right]\right\}\left[\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right]^{n-1}$$

$$= n f_{X_i}(t)\; F_{X_i}^{n-1}(t)$$

$$F_{X_{(1)}}(t) = 1 - \left[1 - F_X(t)\right]^n$$

$$= 1 - \left[1 - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right]^n$$

$$f_{X_1}(t) = \frac{d}{dt} F_{X_{(1)}}(t)$$

$$= -n\left(1 - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx\right)^{n-1} f_{X_i}(t)$$

$$= -n\left(1 - F_{X_i}(t)\right)^{n-1} f_{X_i}(t)$$

NEITHER IS NORMALLY DISTRIBUTED

Pg 222

1) $f(x; v) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{1}{2} \frac{(x-\mu)^2}{v}}$

$L(v) = \prod_{i=1}^{n} f(x; v)$

$= \left(\frac{1}{2\pi v}\right)^{\frac{n}{2}} e^{-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{v}}$

$\ln L(v) = -\frac{n}{2} \ln 2\pi v - \frac{1}{2} \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{v}$

$= -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln v - \frac{1}{2v} \sum_{i=1}^{n} (x_i-\mu)^2$

$\mu = 10$

$\Rightarrow \ln L(v) = -\frac{n}{2} \ln 2\pi - \frac{n}{2} \ln v - \frac{1}{2v} \sum_{i=1}^{n} (x_i - 10)^2$

$\frac{d \ln L(v)}{dv} = -\frac{n}{2v} + \frac{1}{2v^2} \sum (x_i - 10)^2 = 0$

IF $v \neq 0$

$\frac{1}{v} \sum (x_i - 10)^2 = n$

$\Rightarrow v = \frac{\sum_{i=1}^{n} (x_i - 10)^2}{n}$ $(\neq s^2)$

2) $f_{X_i}(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$

WE WISH TO COMPUTE THE MAXIMUM LIKELIHOOD ESTIMATORS OF μ & σ^2 FOR $n=12$

M.L.E. OF $\mu = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{12}\sum_{i=1}^{12} X_i = \frac{66}{12} = 5.5$

M.L.E. OF $\sigma^2 = \hat{\sigma}^2 = \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}\right)^2$

$= \frac{1}{12}(385 - 363) = \frac{11}{6} = 1.83$

3) $f_{x_i}(t) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

WE WISH TO ESTIMATE λ

$$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$= \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod x_i!}$$

$$\ln L(\lambda) = \sum_{i=1}^{n} x_i \ln\lambda - n\lambda - \ln\left[\prod_{i=1}^{n} x_i!\right]$$

$$= \sum_{i=1}^{n} x_i \ln\lambda - n\lambda - \sum_{i=1}^{n} \ln(x_i)!$$

$$\frac{d\ln L(\lambda)}{d\lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{X}$$ IS THE MAXIMUM LIKELIHOOD ESTIMATOR OF λ

4) FROM PROB. 3

THE MAXIMUM LIKELIHOOD ESTIMATOR OF λ IS $\overline{X}$

$$\overline{X} = \frac{30 \text{ CARS}}{20 \text{ DAYS}} = 1.5 \frac{\text{CARS}}{\text{DAY}}$$

$\Rightarrow$ LET $\lambda = 1.5$

5) $P(x) = p(1-p)^{x-1}$

$P(x;p) = p(1-p)^{x-1}$

$$L(p) = \prod_{i=1}^{n} P(x;p) = \prod_{i=n}^{n} \frac{p}{(1-p)} (1-p)^{x_i}$$

$$= \frac{p^n}{(1-p)^n} \prod_{i=1}^{n} (1-p)^{x_i}$$

$$\ln L(p) = n\ln p - n\ln(1-p) + \sum_{i=1}^{n} x_i \ln(1-p)$$

$$= n\ln p - n\ln(1-p) + n\bar{x}\ln(1-p)$$

$$= n\ln p + n(\bar{x}-1)\ln(1-p)$$

$$\frac{d\ln L(p)}{dp} = \frac{n}{p} - n(\bar{x}-1)\frac{1}{1-p} = 0$$

$$\frac{1}{p} = \frac{\bar{x}-1}{1-p}$$

$$p = (1-p)/(\bar{x}-1)$$

$$= \frac{-p}{\bar{x}-1} + \frac{1}{\bar{x}-1}$$

$$p\left(1 + \frac{1}{\bar{x}-1}\right) = \frac{1}{\bar{x}-1}$$

$$p\left(\frac{\bar{x}}{\bar{x}-1}\right) = \frac{1}{\bar{x}-1}$$

$\Rightarrow$ MAXIMUM LIKELIHOOD ESTIMATER OF P IS $1/\bar{x}$

7) $f_X(x) = \lambda e^{-\lambda x} \quad ; \quad x \geq 0$

$f(x;\lambda) = \lambda e^{-\lambda x}$

$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$

$= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

$= \lambda^n e^{-n\lambda \bar{x}}.$

$\ln L(\lambda) = n \ln \lambda - n\lambda \bar{x}$

$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - n\bar{x} = 0 \Rightarrow \lambda = \frac{1}{\bar{x}}$ IS THE MAXIMUM LIKELIHOOD ESTIMATOR OF λ

8) FROM PROB. 7

$\bar{x} = \frac{\sum x_i}{n} = \frac{32,916}{30}$

$\Rightarrow \lambda = \frac{1}{\bar{x}} = \frac{30}{32916} = 9.11 \times 10^{-4}$

9) $P_X(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$

$p(x_i; p) = \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$

$$L(p) = \prod_{i=1}^{N} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$

$$= \left[\prod_{i=1}^{N} \binom{n}{x_i}\right] p^{\sum_{i=1}^{N} x_i} (1-p)^{\sum_{i=1}^{N}(n-x_i)}$$

$$= \left[\prod_{i=1}^{N} \binom{n}{x_i}\right] p^{\sum_{i=1}^{N} x_i} (1-p)^{Nn-\sum_{i=1}^{N} x_i}$$

$$= \left[\prod_{i=1}^{N} \binom{n}{x_i}\right] p^{N\bar{x}} (1-p)^{Nn-N\bar{x}}$$

$$= \left[\prod_{i=1}^{N} \binom{n}{x_i}\right] p^{N\bar{x}} (1-p)^{N(n-\bar{x})}$$

$$\ln L(p) = \ln \prod_{i=1}^{N} \binom{n}{x_i} + N\bar{x} \ln p + N(n-\bar{x}) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{N\bar{x}}{p} - \frac{N(n-\bar{x})}{1-p} = 0$$

$$\frac{\bar{x}}{p} = \frac{n-\bar{x}}{1-p}$$

$$\Rightarrow p\left(\frac{1}{\bar{x}}\right) = \frac{1-p}{n-\bar{x}}$$

$$p\left(\frac{1}{\bar{x}}\right) = \frac{1}{n-\bar{x}} - \frac{p}{n-\bar{x}}$$

$$p\left(\frac{1}{\bar{x}} + \frac{1}{n-\bar{x}}\right) = \frac{1}{n-\bar{x}}$$

$$p \, \frac{n}{\bar{x}(n-\bar{x})} = \frac{1}{n-\bar{x}}$$

$\Rightarrow p = \bar{x}/n$ IS THE MAXIMUM LIKELIHOOD ESTIMATER OF p

SUPPOSE X IS A R.V. WITH $f(x;\theta) = \theta x^{\theta - 1}$ FOR $0 \leq x \leq 1$ AND 0 OTHERWISE AND $\theta > 0$. GIVEN A R.S. OF SIZE n FROM X, FIND THE MAXIMUM LIKELIHOOD ESTIMATER OF θ.

$$f(x;\theta) = \theta x^{\theta-1}$$

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

$$= \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\ln L(\theta) = n \ln\theta + \ln \prod_{i=1}^{n} x_i^{\theta-1}$$

$$= n\ln\theta + \sum_{i=1}^{n} \ln x_i^{\theta-1}$$

$$= n\ln\theta + \sum_{i=1}^{n} (\theta - 1)\ln x_i$$

$$= n\ln\theta + (\theta-1)\sum_{i=1}^{n} \ln x_i$$

$$\frac{d \ln L(\theta)}{d\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i = 0$$

$$\Rightarrow \frac{n}{\theta} = \sum_{i=1}^{n} \ln x_i$$

$$\theta = n \Big/ \sum_{i=1}^{n} \ln x_i$$

IS THE MAXIMUMUM LIKELIHOOD ESTIMATER OF θ

SUPPOSE X IS A GAMMA R.V. WITH

$$f(x;\lambda) = \frac{\lambda(\lambda x)^{r-1} e^{-\lambda x}}{\Gamma(r)} \qquad \text{FOR } x > 0$$

AND SUPPOSE r IS KNOWN. GIVEN A RANDOM SAMPLE SIZE OF n, FIND THE MAXIMUM LIKELIHOOD ESTIMATER OF λ

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{r} x_i^{r-1} e^{-\lambda x_i}}{\Gamma(r)}$$

$$= \frac{\lambda^{rn}}{\Gamma^{n}(r)} e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{r-1}$$

$$= \frac{\lambda^{rn}}{\Gamma^{n}(r)} e^{-\lambda n \bar{x}} \prod_{i=1}^{n} x_i^{r-1}$$

$$\ln L(\lambda) = -n \ln \Gamma(r) + rn \ln \lambda - \lambda n \bar{x} + \ln \prod_{i=1}^{n} x_i^{r-1}$$

$$= -n \ln \Gamma(r) + rn \ln \lambda - \lambda n \bar{x} + \sum_{i=1}^{n} \ln x_i^{r-1}$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{rn}{\lambda} - n\bar{x} = 0$$

$$\Rightarrow \bar{x} = \frac{r}{\lambda}$$

$$\Rightarrow \lambda = \frac{r}{\bar{x}}$$ IS THE MAXIMUM LIKELIHOOD ESTIMATER OF λ

SUPPOSE (JUST SUPPOSE) IS A WEIBUL R.V. WITH $f(x;\lambda) = (\lambda\alpha)\, x^{\alpha-1} e^{\lambda x^{\alpha}}$ FOR $X > 0$, AND SUPPOSE α IS KNOWN. GIVEN A RANDOM SAMPLE OF SIZE n FROM X, FIND THE M.L.E. OF λ

$$L(\lambda) = \prod_{i=1}^{n} f(x_i, \lambda)$$

$$= (\lambda\alpha)^n \prod_{i=1}^{n} x_i^{\alpha-1}\, e^{\lambda \sum_{i=1}^{n} x_i^{\alpha}}$$

$$\ln L(\lambda) = n \ln(\lambda\alpha) + \ln \prod_{i=1}^{n} x_i^{\alpha-1} + \lambda \sum_{i=1}^{n} x_i^{\alpha}$$

$$= n \ln(\lambda\alpha) + \sum_{i=1}^{n} \ln x_i^{\alpha-1} + \lambda \sum_{i=1}^{n} x_i^{\alpha}$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{\alpha} = 0$$

$$\Rightarrow \lambda = -n \Big/ \sum_{i=1}^{n} x_i^{\alpha}$$ IS THE M.L.E. OF λ

Pg 236

1) X_i IS A $N(\mu, \sigma^2)$ RV, $i = 1, 2, 3, 4, 5, 6$

$\Rightarrow (X_1 - X_2), (X_3 - X_4), (X_5 - X)$ ARE ALL $N(0, 2\sigma^2)$ R.V.'s

$\Rightarrow \frac{(X_i - X_2)}{\sqrt{2}\sigma}, \frac{(X_3 - X_4)}{\sqrt{2}\sigma}, \frac{(X_5 - X_6)}{\sqrt{2}\sigma}$ ARE ALL $N(0,1)$ R.V.'s

$\Rightarrow \frac{(X_i - X_2)^2}{2\sigma^2}, \frac{(X_3 - X_4)^2}{2\sigma^2}, \frac{(X_5 - X_6)^2}{2\sigma^2}$ ARE ALL χ^2_1 R.V.'s

LET

$$A = c\left[(X_1 - X_2)^2 + (X_3 - X_4)^2 + (X_5 - X_6)^2\right]$$

$$= c2\sigma^2 \left[\frac{(X_1 - X_2)^2}{2\sigma^2} + \frac{(X_3 - X_4)^2}{2\sigma^2} + \frac{(X_5 - X_6)^2}{2\sigma^2}\right]$$

3 χ^2_1 R.V.'S IS A χ^2_3 R.V.

$\Rightarrow A \sim c2\sigma^2 \chi^2_3$ R.V.

$$E[A] = c2\sigma^2 E[\chi^2_3]$$
$$= 6c\sigma^2$$

IT IS DESIRED THAT $E[A] = \sigma^2 \Rightarrow c = 1/6$

2) $Var(A) = Var\left(\frac{\sigma^2}{3}\chi_3^2\right)$ (SEE PROB 1)

$$= \frac{\sigma^4}{9} Var(\chi_3^2)$$

$$= \frac{\sigma^4}{9}(6) = \frac{2}{3}\sigma^4$$

$$Var(s^2) = \frac{2\sigma^4}{n-1} = \frac{2\sigma^4}{5}$$

$$\Rightarrow Var(s^2) < Var(A)$$

$\therefore$ s^2 IS A BETTER ESTIMATOR

3) LET $A = \frac{1}{n} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2$

X_{2i} é X_{2i-1} ARE $N(\mu, \sigma^2)$ RV

$\Rightarrow X_{2i} - X_{2i-1}$ IS A $N(0, 2\sigma^2)$ RV

$\Rightarrow \frac{X_{2i} - X_{2i-1}}{\sqrt{2}\,\sigma}$ IS A $N(0, 1)$ R.V.

$\Rightarrow \frac{(X_{2i} - X_{2i-1})^2}{2\sigma^2}$ IS A χ^2_1 R.V.

THEN $A = \frac{2\sigma^2}{n} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2 / 2\sigma^2$

$$\begin{aligned} Var\, A &= Var\, \frac{2\sigma^2}{n} \sum_{i=1}^{n/2} \frac{(X_{2i} - X_{2i-1})^2}{2\sigma^2} \\ &= \frac{4\sigma^4}{n^2} Var\, \frac{1}{2\sigma^2} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2 \\ &= \frac{4\sigma^4}{n^2} Var\, \chi^2_{n/2} \\ &= \frac{4\sigma^4}{n^2} (n) = \frac{4\sigma^4}{n} \end{aligned}$$

$\lim_{n \to \infty} Var(A) = 0 \Rightarrow A$ IS A CONSISTANT ESTIMATER OF σ^2

4) $X_{2i} \sim N(\mu, \sigma^2)$ R.V.

$X_{2i} - X_{2i-1} \sim N(0, 2\sigma^2)$ RV

$\dfrac{X_{2i} - X_{2i-1}}{\sqrt{2}\,\sigma} \sim N(0,1)$ R.V.

$\dfrac{(X_{2i} - X_{2i-1})^2}{2\sigma^2} \sim \chi^2_1$ R.V.

$\dfrac{1}{2\sigma^2} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2 \sim \chi^2_{n/2}$ R.V

$\dfrac{1}{n} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2 \sim \dfrac{2\sigma^2}{n} \chi^2_{n/2}$ R.V.

$E\left[\dfrac{2\sigma^2}{n} \chi^2_{n/2}\right] = \dfrac{2\sigma^2}{n} E\left[\chi^2_{n/2}\right]$

$= \sigma^2$

$Var\left[\dfrac{2\sigma^2}{n} \chi^2_{n/2}\right] = \dfrac{4\sigma^4}{n^2} Var\, \chi^2_{n/2}$

$= \dfrac{4\sigma^4}{n}$

$\lim Var\left[\dfrac{2\sigma^2}{n} \chi^2_{n/2}\right] = 0$

$\Rightarrow \dfrac{1}{n} \sum_{i=1}^{n/2} (X_{2i} - X_{2i-1})^2$ IS A CONSISANT ESTIMATER OF σ^2

5) BEST ESTIMATER OF λ IS $\bar{X}$

$\bar{X} = \frac{279}{310}$

$p_X(X) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots$

$p_X(x=0) = e^{-\lambda} \doteq e^{-\bar{X}} = e^{-\frac{279}{310}} = e^{-0.9}$

6) M.L. ESTR. OF λ IS $\overline{X} \Rightarrow \mu = 72.1$

" " " σ^2 IS $\hat{\sigma}^2 = \frac{n-1}{n} s^2 \approx s^2 = 144$

$P[X \geq 50] = P\left[Z \geq \frac{50 - 72.1}{12}\right]$

$= P[Z > -1.842] = 0.9671$

$P[X \leq 90] = P[Z \leq 1.49] = 0.9319$

7) $f_X(x) = \lambda e^{-\lambda x}$

$\mu_X = \frac{1}{\lambda}$; $\sigma_X^2 = \frac{1}{\lambda^2}$

AS $n \to \infty$, Λ, THE MAXIMUM LIKELIHOOD ESTIMATER OF λ, WILL APPROACH A $N\left(\lambda, \frac{1}{n E\left[\left\{\frac{\delta}{\delta\lambda} \ln f(x;\lambda)\right\}^2\right]}\right)$ R.V.

$$f(x;\lambda) = \lambda e^{-\lambda x}$$

$$\ln f(x;\lambda) = \ln \lambda - \lambda x$$

$$\frac{\delta \ln f(x;\lambda)}{\delta \lambda} = \frac{1}{\lambda} - x$$

$$= -(x-\mu)$$

$$E[-(x-\mu)] = -E[(x-\mu)] = 0$$

$$\Rightarrow \Lambda \sim N\left(\frac{1}{\lambda}, 0\right) \text{ R.V.}$$

TEST 1

1. A RANDOM SAMPLE [illegible] DATA

$-1, -2, 0, 1, -3$

COMPUTE THE SAMPLE MEAN, VARIANCE, [illegible]

[illegible]

x_i	x_i^2
-3	9
-2	4
-1	1
0	0
1	1
-5	15

$\bar{x} = \sum_{i=1}^{5} x_i / n = -5/5 = -1$ ⇐ MEAN

$s^2 = \sum x_i^2 - n\bar{x}^2$ ⇐ VARIANCE

$= 15 - 5 = 10$

$s = \sqrt{10}$ ⇐ STANDARD DEVIATION

$m = -1$ ⇐ MEDIAN

$R = 1 + 3 = 4$ ⇐ RANGE

$[X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}]$

$= [-3, -2, -1, 0, 1]$ ⇐ ORDER STATISTICS

[illegible]

OF [illegible] [illegible] [illegible]

TUBES WHOSE OPERATING [illegible] IS AN

EXPONENTIAL RV WITH MEAN [illegible]

OF 2000 HRS. USE THE CENTRAL

LIMIT [illegible] TO APPROXIMATE THE

PROBABILITY THAT THE [illegible]

AVERAGE OF 100 SUCH TUBES

WILL EXCEED 2100 [illegible] (HINT $\sigma = \mu$)

SOL. $n = 100$, $\mu_x = \mu_{\bar{x}} = 2000$, $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{2000}{10} = 200$

$$P[\bar{X} > 2100] = P[\bar{X}_n > 2100]$$

$$= P\left[Z > \frac{2100 - 2000}{200}\right]$$

$$= P\left[Z > \tfrac{1}{2}\right]$$

$$= 1 - P\left[Z < \tfrac{1}{2}\right]$$

$$= 1 - 0.6915$$

$$= 0.3085$$

3) X_1 IS $N(20, 4)$
X_2 IS $N(10, 3)$
X_3 IS $N(15, 6)$
X_1, X_2, X_3 ARE INDEPENDENT R.V

a) WRITE DOWN THE JOINT DENSITY FUNCTION FOR X_1, X_2, AND X_3

$$f_{X_i}(x_i) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2}$$

$$f_Y(y) = \prod_{i=1}^{3} f_{X_i}(x_i)$$

$$= \frac{(2\pi)^{-3/2}}{\sigma_1\sigma_2\sigma_3} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2+\left(\frac{x_3-\mu_3}{\sigma_3}\right)^2\right]}$$

$$= \frac{2\pi^{-3/2}}{2\sqrt{3}\times\sqrt{6}} e^{-\frac{1}{2}\left[\frac{(x_1-20)^2}{4}+\frac{(x_2-10)^2}{3}+\frac{(x_3-15)^2}{6}\right]}$$

b) WRITE DOWN THE DENSITY FUNCTION OF
$$Y = 3X_1 - 2X_2 + 5X_3$$

$$\mu_Y = 3\mu_1 - 2\mu_2 + 5\mu_3$$
$$= 3(20) - 2(10) + 5(15)$$
$$= 60 - 20 + 75 = 115$$
$$\sigma_Y^2 = (3)^2\sigma_1^2 + (2)^2\sigma_2^2 + (5)^2\sigma_3^2$$
$$= (9)(4) + (4)(3) + (25)(6)$$
$$= 36 + 12 + 150 = 198$$
$$\Rightarrow Y \text{ IS } N(115, 198)$$

4) a) PROVE THAT THE MGF OF THE POISSON DISTRIBUTION $P(x) = \frac{1}{x!} e^{-\lambda} \lambda^x \; ; x = 0, 1, 2, \ldots$ IS $e^{\lambda(e^t - 1)}$

$$M_x(t) = E\left[e^{xt}\right]$$

$$= \sum_{x=0}^{\infty} \frac{1}{x!} e^{-\lambda} \lambda^x e^{xt}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{1}{x!} \left[e^t \lambda\right]^x$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{b^x}{x!} \quad ; \; b = e^t \lambda$$

$$= e^{-\lambda} e^{b}$$

$$= e^{-\lambda} e^{e^t \lambda} = e^{\lambda(e^t - 1)}$$

b) USE THE RESULTS OF PART (a) TO PROVE THAT $\mu = E[X] = \lambda$

$$\mu = M_x'(t)\Big|_0$$

$$M_x'(t) = \left[\frac{d}{dt} \lambda e^t\right] e^{\lambda(e^t - 1)}$$

$$= \lambda e^t e^{\lambda(e^t - 1)}$$

$$\mu = M_x'(t)\Big|_0 = \lambda$$

c) USE THE RESULT [illegible] TO FIND THE M.G.F. OF $Z_\lambda = \frac{X-\lambda}{\sqrt{\lambda}} = \frac{X}{\sqrt{\lambda}} - \sqrt{\lambda}$

$Z = aX + b$

$\Rightarrow M_Z(t) = e^{bt} M_X(at)$

$M_X(t) = e^{\lambda(e^t - 1)}$

$\Rightarrow M_X(at) = e^{\lambda(e^{at} - 1)}$

$\Rightarrow M_Z(t) = e^{bt} e^{\lambda(e^{at} - 1)}$

$a = 1/\sqrt{\lambda}$; $b = -\sqrt{\lambda}$

$\therefore M_Z(t) = e^{-\sqrt{\lambda}t} e^{\lambda[e^{t/\sqrt{\lambda}} - 1]}$

d) SHOW [illegible] LIMIT OF $\lambda \to \infty$ OF THE MGF OF Z_λ IN [illegible] $e^{\frac{1}{2}t^2}$

$\ln M_Z(t) = \lambda[e^{t/\sqrt{\lambda}} - 1] - \sqrt{\lambda}t$

$\lim_{\lambda\to\infty} \ln M_Z(t) = [\lambda e^{t/\sqrt{\lambda}} - \lambda - \sqrt{\lambda}t]$

$= [\lambda e^{t\lambda^{-1/2}} - \lambda - \sqrt{\lambda}t]$

$= \lim [\lambda(1 + \frac{t\lambda^{-1/2}}{1} + \frac{(t\lambda^{-1/2})^2}{2!} + \frac{(t\lambda^{-1/2})^3}{3!} + \dots) - \lambda - \sqrt{\lambda}t]$

$= \lim [\lambda(1 + \frac{t}{\sqrt{\lambda}} + \frac{1}{2!}t^2\lambda^{-1} + \frac{1}{3!}t^3\lambda^{-3/2} + \dots) - \lambda - \sqrt{\lambda}t]$

$= \lim [\lambda + t\sqrt{\lambda} + \frac{1}{2!}t^2 + \frac{1}{3!}t^3\lambda^{-1/2} + \dots) - \lambda - \sqrt{\lambda}t]$

$= \frac{1}{2}t^2$

$\Rightarrow \lim_{\lambda\to\infty} M_Z(t) = e^{\frac{1}{2}t^2}$

3) LET $\overline{X}$ BE A POISSON R.V. WITH $\lambda = 16$. USE CHEBYSHEV'S INEQUALITY TO OBTAIN AN UPPER BOUND TO

$$P[\overline{X} \geq 22 \text{ OR } \overline{X} \leq 10]$$

$$P[\overline{X} \geq 22 \text{ OR } \overline{X} \leq 10] = P[\overline{X}-\mu \geq 6 \text{ OR } \overline{X}-\mu \leq -6]$$
$$= P[|\overline{X}-\mu| \leq 6]$$

$$\Rightarrow P[|\overline{X}-\mu| \leq 6] > 1 - 1/k^2$$
$$6 = k\sigma = k\sqrt{\lambda} = k4 \Rightarrow k = \frac{6}{4} = 3/2$$

$$\therefore P[|\overline{X}-\mu| \leq 6] > 1 - \frac{4}{9} = 5/9$$

4) LET $\overline{X}$ BE A POISSON R.V. WITH $\lambda = 16$. USE THE NORMAL APPROX. TO THE POISSON DISTRIBUTION TO FIND

$$P[X \geq 22 \text{ OR } \overline{X} \leq 10]$$
$$= 1 - P[10 < X \leq 22]$$
$$\approx 1 - P[9.5 < X_N < 22.5] \quad \mu = \lambda = 16, \ \sigma = 4$$
$$= P\left[\frac{9.5-16}{4} < Z < \frac{22.5-16}{4}\right]$$
$$= 2P\left[Z > \frac{6.5}{4}\right]$$

Pp. 249-50

1) $X \sim N(\mu, 10^2)$; $n = 10$ $\quad \alpha = 0.05$

$\sum_{i=1}^{10} X_i = 1652$; $\sum_{i=1}^{10} X_i^2 = 273,765$

$1-\alpha$ CONF. INTERVAL

$\bar{X} \pm Z_{\alpha/2}\, \sigma/\sqrt{n}$

OR $\frac{1652}{10} \pm Z_{0.025} \frac{10}{\sqrt{10}}$

$Z_{0.025} = 1.96$

$165.2 \pm (1.96)\sqrt{10}$

OR 165.2 ± 6.2

$(159.0 - 171.4)$

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2) $1-\alpha$ CONF INTERVAL

$\bar{X} \pm t_{n-1;\alpha/2}\ S/\sqrt{n}$

$S = \frac{\sum X_i^2 - n\bar{X}^2}{n-1} = \frac{273765 - 10(165.2)^2}{9} = 95$

$t_{9;0.025} = 2.262$

$165.2 \pm 2.262 \times 95/\sqrt{10} = 165.2 \pm 7$

$\Rightarrow (158.2\ ,\ 172.2]$

3) FIND $L_2 \ni$

$$P\left[\sigma^2 \le L_2\right] = 1-\alpha$$

$$= P\left[\frac{1}{\sigma^2} \ge \frac{1}{L_2}\right]$$

$$= P\left[\frac{(n-1)s^2}{\sigma^2} \ge \frac{(n-1)s^2}{L_2}\right]$$

BUT $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\Rightarrow P\left[\sigma^2 \le L_2\right] = P\left[\chi^2_{n-1} \ge \frac{(n-1)s^2}{L_2}\right] = 1-\alpha$$

$$\therefore \frac{(n-1)s^2}{L_2} = \chi^2_{n-1;1-\alpha}$$

$$\Rightarrow L_2 = \frac{(n-1)s^2}{\chi^2_{n-1;1-\alpha}}$$

4) $P[L_1 \le \sigma \le L_2] = 1-\alpha$

$= P[L_1^2 \le \sigma^2 \le L_2]$

$= P\left[\frac{1}{L_2^2} \le \frac{1}{\sigma^2} \le \frac{1}{L_1^2}\right]$

$= P\left[\frac{(n-1)s^2}{L_2^2} \le \frac{(n-1)s^2}{\sigma^2} \le \frac{(n-1)s^2}{L_2^2}\right]$

$= P\left[\frac{(n-1)s^2}{L_2^2} \le \chi^2_{n-1} \le \frac{(n-1)s^2}{L_2^2}\right]$

WE KNOW

$P[\chi^2_{n-1;\alpha/2} \le \chi^2_{n-1} \le \chi^2_{n-1;1-\alpha/2}] = 1-\alpha$

$\Rightarrow \frac{(n-1)s^2}{L_2^2} = \chi^2_{n-1;\alpha/2}$

OR $L_2 = \sqrt{\frac{n-1}{\chi^2_{n-1;\alpha/2}}}\, s$

$\Rightarrow \frac{(n-1)s^2}{L_1^2} = \chi^2_{n-1;1-\alpha/2}$

OR $L_1 = \sqrt{\frac{n-1}{\chi^2_{n-1;1-\alpha/2}}}\, s$

5) $X \sim N(\mu_X, \sigma^2)$; $Y \sim N(\mu_Y, \sigma^2)$

$\Rightarrow \bar{X} \sim N(\mu_X, \sigma^2/n)$; $\bar{Y} \sim N(\mu_Y, \sigma^2/n)$

$\bar{X} - \bar{Y} \sim N\left(\mu_X - \mu_Y, \frac{2\sigma^2}{n}\right)$

$$\frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{2}{n}}\,\sigma} \sim N(0,1)$$

$$\Rightarrow P\left[-Z_{\alpha/2} \le \frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{2}{n}}\,\sigma} \le Z_{\alpha/2}\right] = 1-\alpha$$

$$= P\left[-\sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2} \le (\bar{X}-\bar{Y}) - (\mu_X - \mu_Y) \le \sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2}\right]$$

$$= P\left[-\sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2} \le (\mu_X - \mu_Y) - (\bar{X}-\bar{Y}) \le \sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2}\right]$$

$$= P\left[(\bar{X}-\bar{Y}) - \sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2} \le \mu_X - \mu_Y \le (\bar{X}-\bar{Y}) + \sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2}\right]$$

$\therefore$ CONFIDENCE INTERVAL

$$(\bar{X}-\bar{Y}) \pm \sqrt{\tfrac{2}{n}}\,\sigma Z_{\alpha/2}$$

5) $\bar{X} - \bar{Y} \sim N\left(\mu_x - \mu_y, \frac{2\sigma^2}{n}\right)$

WE KNOW $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\Rightarrow \frac{(n-1)(S_x^2 + S_y^2)}{\sigma^2} \sim \chi^2_{2(n-1)}$$

AND: $$\frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{2}{n}}\,\sigma} \sim N(0,1)$$

$$t_{2(n-1)} = \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/(n-1)}} = \frac{[(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)]\cancel{\sigma}\sqrt{n-1}\,\cancel{\sqrt{2}}}{\sqrt{\frac{2}{n}}\,\cancel{\sigma}\sqrt{\cancel{(n-1)}}\sqrt{(S_x^2 + S_y^2)}}$$

$$= \sqrt{n}\,\frac{[(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)]}{\sqrt{S_x^2 + S_y^2}}$$

$$P[-t_{2(n-1)\alpha/2} \le t_{2(n-1)} \le t_{2(n-1);\alpha/2}] = 1 - \alpha$$

$$P\left[-t_{2(n-1)\alpha/2} \le \frac{(\bar{X} - \bar{Y}) - (\mu_x - \mu_y)}{\sqrt{\frac{S_x^2 + S_y^2}{n}}} \le t_{2(n-1);\alpha/2}\right] = 1 - \alpha$$

$$= P\left[-\sqrt{\frac{S_x^2 + S_y^2}{n}}\,t_{2(n-1);\alpha/2} \le (\bar{X} - \bar{Y}) - (\mu_x - \mu_y) \le \sqrt{\frac{S_x^2 + S_y^2}{n}}\,t_{2(n-1);\alpha/2}\right]$$

$$= P\left[-\sqrt{\frac{S_x^2 + S_y^2}{n}}\,t_{2(n-1);\alpha/2} \le (\mu_x - \mu_y) - (\bar{X} - \bar{Y}) \le \sqrt{\frac{S_x^2 + S_y^2}{n}}\,t_{2(n-1);\alpha/2}\right]$$

$\Rightarrow$ CONF. INTERVAL

$$(\bar{X} - \bar{Y}) \pm \sqrt{\frac{S_x^2 + S_y^2}{n}}\,t_{2(n-1);\alpha/2}$$

8) $1-\alpha$ CONF INTERVAL: $\alpha = 0.05$

$$\bar{X} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$L = 2 Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = \frac{1}{2}$$

$$\Rightarrow \sqrt{n} = 4 Z_{\alpha/2} \sigma$$

$$n = 16 Z^2_{\alpha/2} \sigma^2$$

$$Z_{\alpha/2} = Z_{0.025} = 1.96$$

a) $\sigma^2 = 16 \Rightarrow n = 16^2 (1.96)^2 = 983.45$

$$\Rightarrow n = 984$$

b) $\sigma^2 = 10^2 \Rightarrow n = 16 \times 10 \times (1.96)^2 = 6146.56$

$$\Rightarrow n = 6147$$

9) $\alpha = 0.1 \;;\; n = 10$

$$\bar{x} \pm t_{n-1;\alpha/2} \frac{s}{\sqrt{n}}$$

$$L = 2t_{n-1;\alpha/2} \frac{s}{\sqrt{n}}$$

$$P[L > 1.17]$$

$$= P\left[2t_{n-1;\alpha/2} \frac{s}{\sqrt{n}} > 1.17\right]$$

$$= P\left[s > \frac{\sqrt{n}(1.17)}{2t_{n-1;\alpha/2}}\right]$$

$$= P\left[s^2 > \frac{n(1.17)^2}{4t^2_{n-1;\alpha/2}}\right]$$

$$= P\left[\frac{s^2(n-1)}{\sigma^2} > \frac{n(n-1)(1.17)^2}{4\sigma^2 t^2_{n-1;\alpha/2}}\right]$$

$$= P\left[\chi^2_{n-1} > \frac{n(n-1)(1.17)^2}{4\sigma^2 t^2_{n-1;\alpha/2}}\right]$$

$$t_{9;0.05} = 1.833$$

$$\frac{n(n-1)(1.17)^2}{4t^2_{n-1;\alpha/2}} = 9.18$$

$$= P\left[\chi^2_{n-1} \geq \frac{9.18}{\sigma^2}\right]$$

a) $\sigma^2 = 1$

$$\Rightarrow P[\chi^2_{n-1} \geq 9.18] \sim 0.45$$

b) $\sigma^2 = 2$

$$\Rightarrow P[\chi^2_{n-1} > 4.59] \sim 0.85$$

Pg. 258

2) $P[\text{TYPE I ERROR}]$

$= P[H_0 \text{ IS REJECTED GIVEN } \mu = \mu_0]$

a) $\alpha = P[\bar{X} > 3.47 \text{ GIVEN } \mu = 3]$

$= P\left[Z > \frac{3.47 - \mu}{\sigma/\sqrt{n}} = \frac{3.47 - 2.0}{3/4}\right] = 0.05$

b) $\alpha = P[\bar{X} < 0.53]$

$= P\left[Z < \frac{0.53 - \mu}{\sigma/\sqrt{n}} = \frac{0.53 - 2.0}{3/4}\right] = 0.05$

c) $\alpha = P[1.952 < \bar{X} < 2.048]$

$= P\left[\frac{1.952 - 2.}{3/4} < Z < \frac{2.048 - 2}{3/4}\right] = 0.05$

$H_1: \mu > 2$; $H_1: \mu < 2$; $H_1: \mu \neq 2$ RESPECTIVELY

3) $\sigma = 10 \qquad \mu = 120$

$H_0: \mu = 120$

$H_1: \mu < 120$

$\alpha = P[\text{TYPE I ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \mu = 120]$

$= P[\bar{X} < 116.08 \text{ GIVEN } \mu = 120]$

$= P\left[Z < \frac{116.08 - 120}{10/5}\right]$

$= P[Z < -1.96]$

$= 0.025$

$C(\mu_0) = P[\text{TYPE II ERROR}]$

$= P[\text{ACCEPTING } H_0 \text{ GIVEN } \mu = \mu_0]$

$= P[\bar{X} < 116.08 \text{ GIVEN } \mu = \mu_0]$

$Q(\mu_0] = 1 - C(\mu_0)$

$= P[Z < 0.54] = 0.7054$

4) $f_X(x) = \lambda e^{-\lambda x}$

$n = 10$

$H_0 : \lambda = 10^{-3}$

$H_1 : \lambda < 10^{-3}$

REJECT H_0 IF $X_{(1)} \leq 5.129$

$F_{X_{(1)}}(t) = 1 - \left[e^{-\lambda t}\right]^n$

$\Rightarrow f_{X_{(1)}}(t) = n\lambda e^{-\lambda t n}$

P[TYPE I ERROR]

$= P$[REJECTING H_0 GIVEN $\lambda = 10^{-3}$]

$= P[X_{(1)} < 5.129$ GIVEN $\lambda = 10^{-3}]$

$$= \int_0^{5.129} n\lambda e^{-\lambda t n}\,dt$$

$$= \int_0^{5.129} (10)(10^{-3})\, e^{-(10)(10)^{-3}t}\,dt$$

$$= \int_0^{5.129} 10^{-2}\, e^{-10^{-2}t}\,dt$$

$$= -e^{-10^{-2}t}\Big|_0^{5.129} = 0.05$$

4) $n = 20$

$H_0 : \sigma^2 = \frac{1}{4}$

$H_1 : \sigma^2 > \frac{1}{4}$

REJECT H_0 IF $\Sigma(X_i - \bar{X})^2 > 7.536$

$P[\text{TYPE I ERROR}]$

$= P[\text{REJECTING } H_0 \text{ GIVEN } \sigma^2 = \frac{1}{4}]$

$= P[\Sigma(X_i - \bar{X})^2 > 7.536 \text{ GIVEN } \sigma^2 = 1/4]$

$= P[s^2 = \frac{\Sigma(X_i - \bar{X})^2}{n-1} > \frac{7.536}{19} \text{ GIVEN } \sigma^2 = 1/4]$

$= P[\frac{(n-1)s^2}{\sigma^2} > \frac{7.536}{\sigma^2} \text{ GIVEN } \sigma^2 = 1/4]$

$= P[\chi^2_{n-1} > 4(7.536)]$

$= P[\chi^2_{n-1} > 30.144] = 0.05$

Part I – MA 505 ([illegible])

1. A random sample yields the data

$-1, -2, 0, 1, -3, 6$

Compute the sample mean, variance, standard deviation, median, range, and observed values of the order statistics.

(25) 2. A random sample of size n is taken from an infinite population with $\mu = 80$ and $\sigma^2 = 64$.

(a) Apply Chebyshev's theorem to find n such that $P(|\bar{X} - 80| \geq \frac{1}{2}) \leq .08$.

(b) Use the Central Limit Theorem to find n such that $P(|\bar{X} - 80| \geq \frac{1}{2})$ is approximately $.08$.

(25) 3. Suppose X is an exponential random variable with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

, where λ is a positive constant.

(a) Show that the moment generating function for X is $M_X(t) = \dfrac{\lambda}{\lambda - t}$

(b) Use the result of (a) to find μ_X and σ_X^2.

(c) Find the moment generating function of $Z = \dfrac{X - \mu}{\sigma}$ where X is the given exponential random variable, using the results of part (b).

(0) (10) 4. Suppose X_1, X_2, and X_3 are independent random variables with means $\mu_1 = 2$, $\mu_2 = -3$ and $\mu_3 = 4$ and variances $\sigma_1^2 = 2$, $\sigma_2^2 = 3$, and $\sigma_3^2 = 4$. Find the mean and variance of

$$Y = 3X_1 - 2X_2 - 4X_3$$

(0) (10) 5. Let X_1 be a Poisson random variable with parameter λ_1 and X_2 be a Poisson random variable with parameter λ_2. Use moment generating functions to find the distribution of $Y = 2X_1 + X_2$.

(The m.g.f. for a Poisson random variable with parameter λ is $\exp[\lambda e^t - \lambda]$)

(0) (10) 6. Let X be a binomial random variable with parameters $n = 64$ and $p = \frac{1}{4}$. Use the normal approximation to the binomial distribution to find $P(14 \le X \le 17)$

BOB MARKS

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1) $-3, -2, -1, 0, 1, 6 \Rightarrow n = 6$

x_i	x_i^2
-3	9
-2	4
-1	1
0	0
1	1
6	36
1	51

15
36
51

$$\bar{X} = \frac{\sum_{i=1}^{6} X_i}{n} = \frac{1}{6} \Leftarrow \text{MEAN}$$

$$\sum X_i^2 = 51$$

$$S^2 = \frac{\sum X_i^2 - n\bar{X}^2}{n-1}$$

$$= \frac{51 - 6\left(\frac{1}{6}\right)^2}{5}$$

$$= \frac{51 - 1/6}{5}$$

$$= \frac{51 - 0.167}{5}$$

$$= 10.38 \Leftarrow \text{VARIANCE}$$

$$S = \sqrt{S^2} = \sqrt{10.38} \Leftarrow \text{STANDARD DEVIATION}$$

$$\text{MEDIAN} = \frac{-1+0}{2} = \frac{-1}{2}$$

$$\text{RANGE} = 6 - (-3) = 9$$

ORDER STATISTICS

$X_{(1)} = -3$

$X_{(2)} = -2$

$X_{(3)} = -1$

$X_{(4)} = 0$

$X_{(5)} = 1$

$X_{(6)} = 6$

2) $\mu = 80 \qquad \sigma^2 = 64$

a) $P[|\bar{X} - \mu| \geq C] \leq \frac{\sigma_{\bar{x}}^2}{C^2} = \frac{\sigma_x^2}{n C^2}$

$C = \frac{1}{2}$

-2

$\frac{\sigma_x^2}{n C^2} = 0.08$

$n = \frac{\sigma_x^2}{C^2 (0.08)} = \frac{64}{(\frac{1}{4})^2 (0.08)}$

$= \frac{64 \times 16}{0.08}$

$= \frac{1024}{0.08}$

UPPER BOUND $\Rightarrow n = 1.28 \times 10^5$

$\begin{array}{r} 64 \\ 16 \\ \hline 384 \\ 64 \\ \hline 1024 \end{array}$

$.08 \overline{)1024000}$ $\quad 128000$

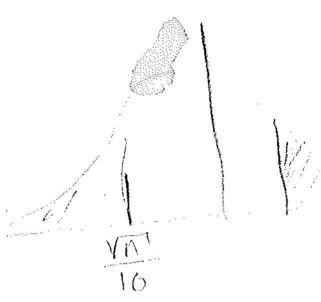
b) $P[|\bar{x} - 80| > \frac{1}{2}] = P\left[\left|\frac{\bar{x} - 80}{\sigma_x/\sqrt{n}}\right| > \frac{\sqrt{n}}{2\sigma_x}\right]$

$\simeq P\left[|Z| > \frac{\sqrt{n}}{2\sigma_x}\right]$

$= P\left[|Z| > \frac{\sqrt{n}}{2 \times 8}\right]$

$= P\left[|Z| > \frac{\sqrt{n}}{16}\right] = 0.08$

$P\left[Z > \frac{\sqrt{n}}{16}\right] = 1 - P\left[Z < \frac{\sqrt{n}}{16}\right]$



$\Rightarrow 0.08 = 2 - 2P\left[Z < \frac{\sqrt{n}}{16}\right]$

$1.92 = 2P\left[Z < \frac{\sqrt{n}}{16}\right]$

$0.810 = P\left[Z < \frac{\sqrt{n}}{16}\right]$

FIND $\psi \ni P[Z < \psi] = 0.81$

$\psi = \frac{\sqrt{n}}{16}$, SOLVE FOR n $(= (16\psi)^2)$

ROUND TO CLOSEST INTEGER

3) a) $M_X(t) = E[e^{xt}]$

$$= \int_0^\infty \lambda e^{-\lambda x} e^{xt} dx$$

$$= \lambda \int_0^\infty e^{x(t-\lambda)} dx$$

$$= \lambda \frac{1}{t-\lambda} e^{x(t-\lambda)} \Big|_0^\infty$$

$$= \frac{\lambda}{\lambda - t} \quad ; t \leq \lambda$$

b) $m_1 = M_x'(0)$

$$M_x(t) = \lambda(\lambda - t)^{-1}$$

$$M_x'(t) = \lambda(-1)(-1)(\lambda - t)^{-2}$$

$$= \frac{\lambda}{(\lambda - t)^2}$$

$$\Rightarrow m_1 = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$M_x'(t) = \lambda(\lambda - t)^{-2}$$

$$M_x''(t) = \lambda(-1)(-2)(\lambda - t)^{-3}$$

$$= \frac{2\lambda}{(\lambda - t)^3}$$

$$m_2 = M_x''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\sigma_x^2 = m_2 - m_1^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

c) $M_X(t) = \frac{\lambda}{\lambda - t}$

$\cancel{Z} = aX + b$

$$\Rightarrow M_{\cancel{Z}}(t) = e^{bt} M_X(at)$$

$$= e^{bt} \frac{\lambda}{\lambda - at}$$

$$a = \frac{1}{\sigma_x} \quad ; \quad b = \frac{-\mu_x}{\sigma_x}$$

$$\Rightarrow M_Z(t) = e^{-\frac{\mu_x}{\sigma_x}t} \frac{\lambda}{\lambda + \frac{\mu_x}{\sigma_x}t}$$

$$\mu_x = \sigma_x = 1/\lambda$$

$$\Rightarrow M_Z(t) = e^{-t} \frac{\lambda}{\lambda + t}$$

$$z = \frac{x - \frac{1}{\lambda}}{\frac{1}{\lambda}} = \lambda x - 1 = -1 + \lambda x$$

$$M_Z(t) = e^{-t} \left(\frac{\lambda}{\lambda - \lambda t} \right) = e^{-t} \frac{1}{1-t}$$

-3

4) $f_{X_i}(x_i) = \frac{1}{\sigma_i\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2}$

$\mu_Y = \sum_{i=1}^{3} a_i \mu_i$

$= (2)(3) + (-3)(-2) + (4)(-4)$

$= 6 + 6 - 16 = 6 - 10 = -4$

$\sigma_Y^2 = \sum a_i^2 \sigma_i^2$

$= (3)^2(2) + (-2)^2(3) + (-4)^2(4)$

$= 9 \times 2 + 4 \times 3 + 16 \times 4$

$= 18 + 12 + 64 = 94$

16
4
74
12
18
94

5) $M_X(t) = \lambda e^{\lambda e^t - \lambda}$

$M_{X_1}(t) = \lambda_1 e^{\lambda_1 e^t - \lambda_1}$

$M_{X_2}(t) = \lambda_2 e^{\lambda_2 e^t - \lambda_2}$

$Y = 2X_1 + X_2$

$M_Y(t) = E[e^{Yt}]$

$= E[e^{(2X_1 + X_2)t}]$

$= E[e^{2X_1 t} e^{X_2 t}]$

$= E[e^{2X_1 t}] E[e^{X_2 t}]$

$E[e^{X_1(2t)}] = \lambda_1 e^{(\lambda_1 e^{2t} - \lambda_1)}$

$E[e^{X_2 t}] = M_{X_2}(t)$

$\Rightarrow M_Y(t) = \cancel{\lambda_1} e^{(\lambda_1 e^{2t} - \lambda_1)} \cancel{\lambda_2} e^{(\lambda_2 e^t - \lambda_2)}$

-3

$$M_Y(t) = \lambda_1\lambda_2 e^{(\lambda_1 e^{2t}-\lambda_1)} e^{(\lambda_2 e^{t}-\lambda_2)}$$
$$= \lambda_1\lambda_2 e^{(\lambda_1 e^{2t}+\lambda_2 e^{t})-(\lambda_1+\lambda_2)}$$
$$= \lambda_1\lambda_2 e^{\lambda_1(e^{2t}-1)} e^{\lambda_2(e^{t}-1)}$$
$$= \lambda_1\lambda_2 e$$

6) $P[14 \le X \le 17]$

$\mu = np = \frac{1}{4}64 = 16$

$\sigma^2 = \mu(1-p) = 16\left(\frac{3}{4}\right) = 12$

$\Rightarrow \sigma = \sqrt{12}$

$$P[14 \le X \le 17] = P\left[\frac{14-16}{\sqrt{12}} \le \frac{X-\mu}{\sigma} \le \frac{17-16}{\sqrt{12}}\right]$$
$$= P\left[\frac{-2}{\sqrt{12}} \le \frac{X-\mu}{\sigma} \le \frac{1}{\sqrt{12}}\right]$$
$$\simeq P\left[\frac{-2}{\sqrt{12}} - \cancel{\frac{1}{2}} \le Z \le \frac{1}{\sqrt{12}} + \cancel{\frac{1}{2}}\right]$$

-3

Test 2 — MATH 505

(30) 1. $X_1, X_2, \cdots, X_n$ are a random sample of X, where $f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) Use the maximum likelihood method to find an estimator $\hat{\lambda}$ of λ.

(b) If n is large, find the approximate <u>mean</u> and <u>variance</u> of $\hat{\lambda}$. Show the work leading to your answer for the variance.

(c) If n is large, what is the approximate distribution of $\hat{\lambda}$?

(Hint: $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$)

(15) 2. The mean and variance of a χ^2_ν random variable are ν and 2ν, respectively. Use these results to find the <u>mean</u> and <u>variance</u> of $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$, if we take a random sample of n from a $N(\mu, \sigma^2)$ population.

7) 3. In Problem 1, using the result of 1(a), find the maximum likelihood estimator of $P(X \leq 2)$.

(7) 4. $X_1, X_2, \ldots, X_n$ is a random sample of the random variable X, where

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that the density function of $X_{(n)}$, the largest sample value, is

$$f_{X_{(n)}}(t) = \begin{cases} \frac{n t^{n-1}}{\theta^n} & \text{for } 0 \le t \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(b) Check whether $X_{(n)}$ is an unbiased estimator of θ.

(c) Check whether $X_{(n)}$ is a consistent estimator of θ.

(d) Find the density function of $X_{(1)}$, the smallest sample value.

5. Given a random sample of size n from a
(8) population having density function

$$f(x) = \begin{cases} e^{-(x-\theta)} & \text{for } x \ge \theta \\ 0 & \text{otherwise} \end{cases},$$

find the maximum likelihood estimator of θ.

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1) a) $L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$

$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$

$= \lambda^n e^{-\lambda n \bar{x}}$

$\ln L(\lambda) = n \ln \lambda - \lambda n \bar{x}$

$\frac{\delta \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - n\bar{x} = 0 \Rightarrow \lambda = \frac{1}{\bar{x}}$

SO THAT $\hat{\lambda} = 1/\bar{x}$ IS THE M.L.E. OF λ

b) $\hat{\lambda}$ IS A $N(\mu, \sigma^2)$ R.V. WITH

$\mu = \lambda$

AND $\sigma^2 \cong \frac{1}{nE\left[\left\{\frac{d}{\delta\lambda}\ln f(x;\lambda)\right\}^2\right]}$; FOR n LARGE

$f(x,\lambda) = \lambda e^{-\lambda x}$

$\ln f(x,\lambda) = \ln \lambda - \lambda x$

$\frac{d \ln f(x,\lambda)}{\delta\lambda} = \frac{1}{\lambda} - x$

$\left[\frac{\delta \ln f(x,\lambda)}{\delta\lambda}\right]^2 = \left[x - \frac{1}{\lambda}\right]^2$

$= [x - \mu]^2$

$E[(x-\mu)^2] = \sigma^2 = \cancel{\frac{1}{\lambda^2}}$

$\Rightarrow \frac{1}{nE\left[\left\{\frac{\delta}{\delta\lambda}\ln f(x;\lambda)\right\}\right]^2} = \frac{\lambda^2}{n} = Var(\hat{\lambda})$

c) $\therefore \hat{\lambda}$ IS A $N\left(\lambda, \frac{\lambda^2}{n}\right)$ R.V.

2) $S^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$

WE KNOW

~~$S^2 \sim$~~

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$Var\left[\frac{(n-1)S^2}{\sigma^2}\right] = Var\left[\chi^2_{n-1}\right]$

$\frac{(n-1)^2}{\sigma^4} Var(S^2) \quad = 2(n-1)$

$\Rightarrow Var(S^2) = \frac{2\sigma^4}{n-1}$

$E\left[\frac{(n-1)S^2}{\sigma^2}\right] = E\left[\chi^2_{n-1}\right]$

$\frac{n-1}{\sigma^2} E\left[S^2\right] = n-1$

$\Rightarrow E\left[S^2\right] = \sigma^2$

$\therefore S^2$ HAS MEAN σ^2 AND VARIANCE OF $\frac{2\sigma^4}{n-1}$

3) $\cancel{P[X \geq 2] = P[X=2] + P[X=1]}$

$$\cancel{P[X \le 2] = P[X=2] + P[X=1] + P[X=0]}$$

$$\cancel{= \lambda e^{-\lambda 2}}$$

$$P[X \le 2] = \int_0^2 \lambda e^{-\lambda x}$$

$$= -e^{-\lambda x}\Big|_0^2$$

$$= -e^{-\lambda 2} + 1$$

$$= 1 - e^{-2\lambda}$$

THE M.L.E. OF λ IS $\frac{1}{\overline{X}}$

$\Rightarrow$ THE M.L.E OF $P[X \le 2]$

$$= 1 - e^{-2/\overline{X}}$$

(4) a) $F_{X_{(n)}}(t) = \left[F_X(t)\right]^n$

$$F_X = \int_0^t \frac{1}{\theta}\, dx = \frac{t}{\theta} \quad \text{FOR } 0 < t < \theta$$

$$\Rightarrow F_{X_{(n)}}(t) = \left[\frac{t}{\theta}\right]^n$$

$$f_{X_{(n)}}(t) = \frac{d}{dt} F_{X_{(n)}}(t)$$

$$= \frac{d}{dt} \frac{t^n}{\theta^n}$$

$$= \begin{cases} \dfrac{n t^{n-1}}{\theta^n} & \text{FOR } 0 \le t \le \theta \\ 0 & \text{OTHERWISE} \end{cases}$$

b) $$E\left[X_{(n)}\right] = \int_0^\theta \frac{n x^n}{\theta^n}\, dx$$

$$= \frac{n\, x^{n+1}}{(n+1)\,\theta^n}\Big|_0^\theta$$

$$= \frac{n\, \theta^{n+1}}{(n+1)\,\theta^n} = \frac{n}{n+1}\theta \neq \text{NOT UNBIASED}$$

c) ~~$\lim E[X_n$~~

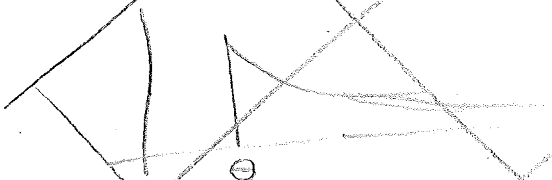
~~$F_{X_{(n)}} = \left[F_X(t)\right]^n$~~

d) $F_{X_{(1)}}(t) = 1 - [1 - F_X(t)]^n$

$\Rightarrow f_{X_{(1)}}(t) = \frac{d}{dt} F_{X_{(1)}}(t) = n[1 - F_X(t)]^{n-1} f_X(t)$

$= \begin{cases} \left[1 - \frac{t}{\theta}\right]^{n-1} \frac{n}{\theta} & ; \text{FOR } 0 \le X \le \theta \\ 0 & ; \text{OTHERWISE} \end{cases}$

5) $f(\theta; x) = e^{-x+\theta} \quad ; x > \theta$



$\int_\theta^\infty e^{-x+\theta}\,dx$
$-e^{-x+\theta}\Big|_\theta^\infty$
$0 + e^{0}$

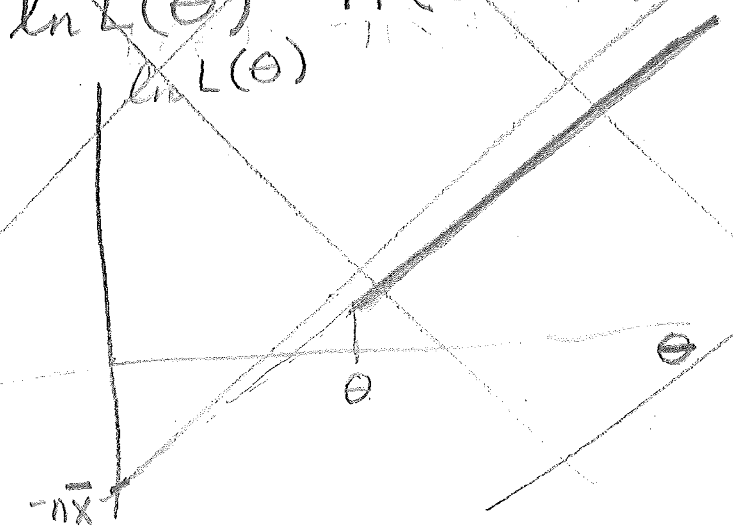
$L(\theta) = \prod_{i=1}^{n} e^{-x_i+\theta}$

$= e^{-\sum_{i=1}^{n} x_i} e^{n\theta}$

$= e^{-n\bar{X}} e^{n\theta} \quad x > \theta$

$= e^{n(\theta - \bar{X})} \quad x > \theta$

$\ln L(\theta) = n(\theta - \bar{X}) = n\theta - n\bar{X} \quad x > \theta$



$\ln L(\theta)$ ~~IS~~ HAS ~~MAX. FOR~~ NO RELATIVE MAXIMA.

$\Rightarrow L(\theta)$ HAS NO RELATIVE MAXIMA.

$\Rightarrow$ THE LARGER θ THE LARGER $L(\theta)$

$$f(x) = e^{-(x-\Theta)} \quad \text{FOR } \cancel{x-\Theta} \geq \Theta$$

LET $\psi = X - \Theta$ $\qquad \psi \geq 0$

$$f(x) = e^{-\psi_0}$$

$$L(\psi) = \prod_{i=1}^{n} e^{-\psi_i} \qquad \psi_i \geq 0 \text{ for } \underline{\underline{all}} \; x_i$$

$$= e^{-\sum_{i=1}^{n} \psi_i}$$

$$= e^{-n\bar{\psi}} \qquad \psi \geq 0$$

$$\psi > 0 \Rightarrow \bar{\psi} > 0$$

$L(\psi) = e^{-n\bar{\psi}}$ IS MAXIMUM

AT $-n\bar{\psi} = 0$ OR AT

$$\sum_{i=1}^{n} (X_i - \Theta) = 0 = \sum_{i=1}^{n} X_i - n\Theta$$

-4

$$\Rightarrow n\Theta = \sum_{i=1}^{n} X_i$$

$$\Theta = \bar{X}$$

$\therefore \bar{X}$ IS THE MAXIMUM LIKELIHOOD ESTIMATER OF Θ, MATH 505 - Test 2

1. (a) $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$

$$L(\lambda) = \prod_{i=1}^{n} f(x_i; \lambda) = \lambda^n e^{-\lambda \sum x_i}$$

$$\log L(\lambda) = n \log \lambda - \lambda \sum x_i$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{n}{\lambda} - \sum x_i = 0$$

$$\lambda = \frac{n}{\sum x_i} = \frac{1}{\bar{x}} = \hat{\Lambda}$$

(b) $\lim_{n \to \infty} E(\hat{\Lambda}) = \lambda$

$$\mathrm{Var}(\hat{\Lambda}) \to \frac{1}{n E\left\{\left[\frac{\partial}{\partial \lambda} \ln f(x; \lambda)\right]^2\right\}}$$

$$f(x; \lambda) = \lambda e^{-\lambda x}$$

$$\ln f(x; \lambda) = \ln \lambda - \lambda x$$

$$\frac{\partial}{\partial \lambda} \ln f(x; \lambda) = \frac{1}{\lambda} - x$$

$$\left[\frac{\partial}{\partial \lambda} \ln f(x; \lambda)\right]^2 = \left(\frac{1}{\lambda} - x\right)^2 = \left(x - \frac{1}{\lambda}\right)^2$$

$$E\left\{\left[\frac{\partial}{\partial \lambda} \ln f(x; \lambda)\right]^2\right\} = E\left[\left(X - \frac{1}{\lambda}\right)^2\right] = \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

$$V(\hat{\Lambda}) \to \frac{1}{n \frac{1}{\lambda^2}} = \frac{\lambda^2}{n}$$

(c) $V(\hat{\Lambda}) = \lambda^2/n$

$$E(\chi^2_{n-1}) = E\left[\frac{(n-1)S^2}{\sigma^2}\right]$$

$$n-1 = \frac{n-1}{\sigma^2} E(S^2)$$

$$E(S^2) = \sigma^2$$

$$Var(\chi^2_{n-1}) = Var\left[\frac{(n-1)S^2}{\sigma^2}\right]$$

$$2(n-1) = \frac{(n-1)^2}{\sigma^4} Var(S^2)$$

$$Var(S^2) = \frac{2\sigma^4}{n-1}$$

3. $$P(X \le 2) = \int_0^2 \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^2$$

$$= 1 - e^{-2\lambda}$$

estimator of $P(X \le 2)$ is $1 - e^{-2/\bar{X}}$

4(a) $$f_{(X_n)}(x) = n\left[F_X(x)\right]^{n-1} f_X(x)$$

where $f_X(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$ and $F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x}{\theta} & \text{for } 0 \le x < \theta \\ 1 & \text{for } x \ge \theta \end{cases}$

$$f_{X_{(n)}}(x) = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta} = \begin{cases} \frac{n}{\theta^n} x^{n-1} & \text{for } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(b) $E[X_{(n)}] = \int_0^\theta x \frac{n}{\theta^n} x^{n-1}\,dx = \frac{n}{\theta^n}\int_0^\theta x^{n}\,dx$

$$= \frac{n}{\theta^n}\,\frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\,\theta \quad \text{— biased}$$

(c) $E[X_{(n)}^2] = \int_0^\theta x^2 \frac{n}{\theta^n} x^{n-1}\,dx = \frac{n}{\theta^n}\int_0^\theta x^{n+1}\,dx$

$$= \frac{n}{\theta^n}\,\frac{x^{n+2}}{n+2}\Big|_0^\theta = \frac{n\,\theta^2}{n+2}$$

$$\mathrm{Var}[X_{(n)}] = \frac{n}{n+2}\theta^2 - \left[\frac{n}{n+1}\theta\right]^2 = \frac{n}{(n+2)(n+1)^2}\theta^2$$

$\lim_{n\to\infty} E(X_{(n)}) = \theta$ and $\lim_{n\to\infty} \mathrm{Var}[X_{(n)}] = 0$, then

$\therefore$ $X_{(n)}$ is consistent

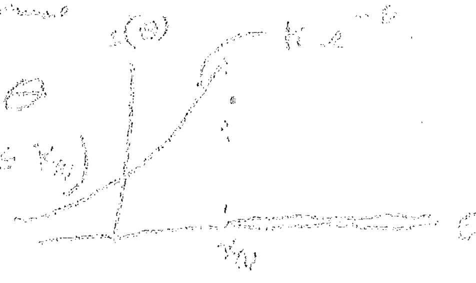
(d) $$f_{X_{(1)}}(x) = n\left[1 - F_X(x)\right]^{n-1} f_X(x)$$

$$= \begin{cases} n\left[1 - \frac{x}{\theta}\right]^{n-1}\frac{1}{\theta} & \text{for } 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

5. $$f(x) = \begin{cases} e^{-(x-\theta)} & \text{for } x \ge \theta \\ 0 & \text{otherwise} \end{cases}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \begin{cases} e^{-\Sigma x_i + n\theta} & \text{for } x_1 \ge \theta,\ x_2 \ge \theta,\ \ldots\ x_n \ge \theta \\ 0 & \text{otherwise} \end{cases}$$

$$L(\theta) = \begin{cases} e^{-\Sigma x_i}\, e^{n\theta} & \text{for } X_{(1)} \ge \theta \\ 0 & \text{otherwise } (\text{or } \theta \le X_{(1)}) \end{cases}$$



m.l.e is $X_{(1)}$.

STAT - MATH 505

( ) 1. Let X and Y be independent normal random variables with unknown means μ_X and μ_Y and unknown variances σ_X^2 and σ_Y^2. However, it is known that $\sigma_X^2 = \sigma_Y^2 = \sigma^2$. A random sample of n is taken from X and a random sample of m $(m \neq n)$ is taken from Y yielding means $\bar{x}$, $\bar{y}$ and variances s_x^2 and s_y^2.

(a) Show that $\dfrac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}}$ is standardized normal random variable.

(b) Show that $\dfrac{(n-1)S_X^2 + (m-1)S_Y^2}{\sigma^2}$ is a χ^2_{n+m-2} random variable.

(c) Use the results of (a) and (b) and the fact that $\bar{X}$, $\bar{Y}$, S_X^2, and S_Y^2 are independent to prove that $\dfrac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}}$

$\left(\text{where } S^2 = \dfrac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}\right)$ is a Student's t random variable with $n+m-2$ degrees of freedom.

(d) Use the results of (c) to derive a two sided $1-\alpha$ confidence interval

( ) for $\mu_X - \mu_Y$.

(29) 2. An urn contains 6 marbles, of which θ are white while the others are black. In order to test $H_0: \theta = 3$ against $H_1: \theta > 3$, two marbles are drawn at random without replacement and H_0 is rejected if both marbles are white. Otherwise H_0 is accepted.

(a) Find the probability of Type I error.

(b) Find the probability of Type II error when $\theta = 5$.

(c) Find the power of this test when $\theta = 5$.

(30) 3. A random sample of n is taken from a random variable whose density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Use the Neyman-Pearson Lemma to show that the most powerful critical (or rejection) region for testing $H_0: \lambda = \lambda_0$ against $H_1: \lambda$ $(\lambda_1 < \lambda_0)$ is to reject H_0 when $\bar{X} > C$, where

( ) C is a constant.

Q7. A random sample of n is taken from a $N(\mu, V)$ population where both μ and V are unknown. Prove that the likelihood ratio test criterion for testing H_0: $V = V_0$ against H_1: $V \neq V_0$ can be written as "Reject H_0 if

$$\lambda = \left(\frac{\hat{V}}{V_0}\right)^{\frac{n}{2}} e^{-\frac{1}{2} n \frac{\hat{V}}{V_0}} e^{\frac{1}{2} n} < A,$$

where A is a constant and $\hat{V} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n}$.

Use any maximum likelihood estimators you need without deriving them.

BOB MARKS

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1)a) $\bar{X} \sim N(\mu_X, \sigma^2/n)$ $\quad\quad \bar{Y} \sim N(\mu_Y, \sigma^2/m)$

$$\Rightarrow \bar{X} - \bar{Y} \sim N\left(\mu_X - \mu_Y, \frac{\sigma^2}{n} + \frac{\sigma^2}{m}\right)$$
$$\sim N\left(\mu_X - \mu_Y, \sigma^2\left(\frac{1}{n} + \frac{1}{m}\right)\right)$$

$$\Rightarrow \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim N(0,1)$$

2) $$\frac{(n-1)S_X^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{(m-1)S_Y^2}{\sigma^2} \sim \chi^2_{m-1}$$

SINCE $\chi^2_{r_1} + \chi^2_{r_2} = \chi^2_{r_1+r_2}$

$$\frac{(n-1)S_X^2 + (m-1)S_Y^2}{\sigma^2} \sim \chi^2_{(n-1)+(m-1)}$$
$$\sim \chi^2_{n+m-2}$$

3) $$T_{n+m-2} = \frac{N(0,1)}{\sqrt{\frac{\chi^2_{n+m-2}}{n+m-2}}}$$

$$= \frac{\left[(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)\right] \cancel{\sigma} \sqrt{n+m-2}}{\cancel{\sigma}\sqrt{\frac{1}{n} + \frac{1}{m}}\,\sqrt{(n-1)S_X^2 + (m-1)S_Y^2}}$$

$\frac{1}{S}$

$$= \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}}$$

$$\Rightarrow S^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{m+n-2}$$

d)

$$\cancel{\bar{X} \pm t_{n-1} \frac{s}{\sqrt{n}}}$$

$$(\bar{X} - \bar{Y}) \pm t_{n+m-2} \frac{s}{\ }$$

$$P\left[-t_{n+m-2;\frac{\alpha}{2}} \le \frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{s\sqrt{\frac{1}{n}+\frac{1}{m}}} \le t_{n+m-2;\alpha/2}\right] = 1-\alpha$$

$$= P\left[-t_{n+m-2;\alpha/2} \le \frac{(\mu_X - \mu_Y) - (\bar{X}-\bar{Y})}{s\sqrt{\frac{1}{n}+\frac{1}{m}}} \le t_{n+m-2;\alpha/2}\right]$$

$$= P\Big[(\bar{X}-\bar{Y}) - s\sqrt{\frac{1}{n}+\frac{1}{m}}\, t_{n+m-2;\alpha/2} \le (\mu_X - \mu_Y) \le (\bar{X}-\bar{Y}) + s\sqrt{\frac{1}{m}+\frac{1}{n}}\, t_{n+m-2;\alpha/2}\Big]$$

$1-\alpha$ CONF. INTERVAL FOR $\mu_X - \mu_Y$ IS

$$(\bar{X}-\bar{Y}) \pm s\sqrt{\frac{1}{m}+\frac{1}{n}}\, t_{n+m-2;\alpha/2}$$

2) $n = 6$

$H_0: \theta = 3$

$H_1: \theta > 3$

REJECT H_0 IF $\sum_{i=1}^{2} X_i = 0$

$\alpha = P[$TYPE I ERROR$]$

$= P[$REJECTING H_0 GIVEN THAT $\theta = 3]$

$= P[$FIRST MARBLE IS ~~[illegible]~~ WHITE

AND SECOND " "

GIVEN $\theta = 3]$

$\cancel{= \left(\frac{1}{2}\right)\left(\right.}$

$= \left(\frac{3}{6}\right)\left(\frac{2}{5}\right) = \frac{1}{5}$

$\beta\big|_{\theta=5} = P[$accepting H_0 given $\theta = 5]$

$= P[$FIRST MARBLE IS BLACK $\theta = 5$

OR SECOND MARBLE IS BLACK$]$

-6

$= \cancel{\frac{1}{6}}$

$\Rightarrow C(5) = \frac{1}{6}$

$\oplus\ Q(5) = 1 - C(5) = 5/6$

3) $f_X(x) = \lambda e^{-\lambda x} \quad \forall x \geq 0$

$H_0: \lambda = \lambda_0 \quad ; \quad H_1\ \lambda = \lambda_1 \qquad \lambda_1 < \lambda_0$

$$L(\vec{x}, \lambda_1) = \prod_{i=1}^{n} \lambda_1 e^{-\lambda_1 x_i}$$

$$= \lambda_1^n e^{-\sum_{i=1}^{n} \lambda_i x_i}$$

$$= \lambda_1^n e^{-n\lambda_1 \bar{x}}$$

$$\frac{L(\vec{x}, \lambda_0)}{L(\vec{x}, \lambda_1)} = \frac{\lambda_0^n e^{-n\lambda_0 \bar{x}}}{\lambda_1^n e^{-n\lambda_1 \bar{x}}} < K$$

$$\Rightarrow e^{-n\lambda_0 \bar{x} + n\lambda_1 \bar{x}} < K'$$

$$-n\lambda_0 \bar{x} + n\lambda_1 \bar{x} < K''$$

$$n\bar{x}(\lambda_1 - \lambda_0) < K''$$

$$\lambda_1 < \lambda_0 \Rightarrow \lambda_1 - \lambda_0 < 0$$

$$\Rightarrow n\bar{x} > K''/(\lambda_1 - \lambda_0) = K'''$$

$$\bar{x} > \frac{K'''}{n} = c$$

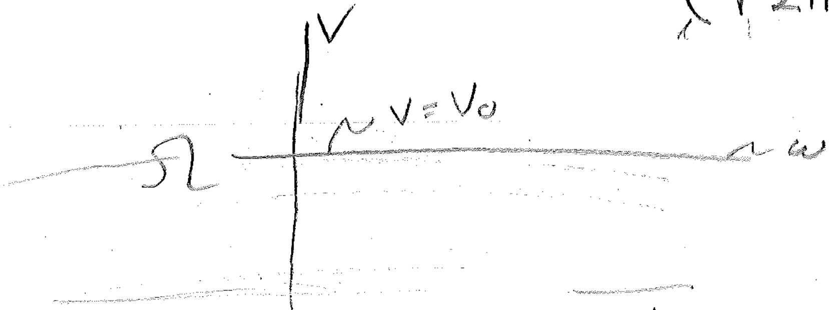
$\Rightarrow$ REJECT H_0 WHEN $\bar{x} > c$

Bob Marks

4) $H_0 : v = v_0 \qquad H_1 : v \neq v_0$

$$L(v, \mu, \vec{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\, v^{1/2}} e^{-\frac{1}{2}\frac{(x_i-\mu)^2}{v}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\, v^{\frac{1}{2}}}\right)^n e^{-\frac{1}{2}\frac{\Sigma (x_i-\mu)^2}{v}}$$



$$\hat{\Omega} = \Big(\overset{\mu}{\bar{x}}, \overbrace{\frac{\Sigma (x_i - \bar{x})^2}{n}}^{\hat{v}}\Big) \quad -5$$

$$\hat{\omega} = (\bar{x}, v_0)$$

$$\frac{L(\hat{\omega})}{L(\hat{\Omega})} = \frac{\left(\frac{1}{\sqrt{2\pi}\, v_0^{1/2}}\right)^n e^{-\frac{1}{2}\frac{\Sigma (x_i - \overset{\bar{x}}{\mu})^2}{v_0}}}{\left(\frac{1}{\sqrt{2\pi}}\, \hat{v}^{1/2}\right)^n e^{-\frac{1}{2}\frac{\Sigma (x_i - \overset{\bar{x}}{\mu})^2}{\hat{v}}}} = \lambda$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[\frac{\Sigma (x_i-\mu)^2}{v_0} - \frac{\Sigma (x_i-\mu)^2}{\hat{v}}\right]}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\Sigma (x_i-\mu)^2\left(\frac{1}{v_0} - \frac{1}{\hat{v}}\right)}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[\Sigma x_i^2 - 2\mu \Sigma x_i + n\mu^2\right]\left(\frac{1}{v_0} - \frac{1}{\hat{v}}\right)}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[\Sigma x_i^2 - 2\mu n\bar{x} + n\mu^2\right]\left[\frac{1}{v_0} - \frac{1}{\hat{v}}\right]}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[\Sigma x_i^2 - n\bar{x}^2 - 2\mu n\bar{x} + n\bar{x}^2 + n\mu^2\right]\left[\frac{1}{v_0} - \frac{1}{\hat{v}}\right]}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[n\hat{v} + n(\bar{x}^2 - 2\mu\bar{x} + \mu^2)\right]\left(\frac{1}{v_0} - \frac{1}{\hat{v}}\right)}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}\left[n\hat{v} + n(\bar{x} - \mu)^2\right]\left(\frac{1}{v_0} - \frac{1}{\hat{v}}\right)}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}n\left(\frac{\hat{v}}{v_0}\right)} e^{\frac{1}{2}n}$$

$$= \left(\frac{\hat{v}}{v_0}\right)^{n/2} e^{-\frac{1}{2}n\left(\frac{\hat{v}}{v_0}\right)} e^{\frac{1}{2}n}$$

TEST # 1 PLUGS TO KNOW AND LOVE # 1

I) RANDOM VARIABLES

A) BINOMIAL: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$; $x = 0, 1, 2, 3, \ldots, n$

$\mu = np$; $\sigma^2 = np(1-p)$

B) POISSON: $p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$; $x = 0, 1, 2, \ldots$

$\mu = \lambda$; $\sigma^2 = \lambda$

C) NORMAL: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

D) EXPODENTIAL: $f_X(x) = \theta e^{-\theta}$

$\mu = \frac{1}{\theta}$; $\sigma^2 = \frac{1}{\theta^2}$

II) MOMENTS AND EXPECTED VALUES

A) $m_n = E[x^n \cancel{f_X(x)}] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx \Leftarrow n^{TH}$ MOMENT

B) MOMENT GENERATING FUNCTION

1) $M_X(t) = E[e^{tx} f_X(x)] = \int_{-\infty}^{\infty} e^{-tx} f_X(x)\,dx$

2) $m_n = \left.\frac{d^n M_X(t)}{dt}\right|_{t=0}$

3) $M_X(t) = \sum_{n=0}^{\infty} \frac{m_n t^n}{n!}$

4) IF $Y = aX + b$

THEN $M_Y(t) = e^{bt} M_X(at$

a) FOR N.R.V. $M_Z(t) = M_{\frac{x-\mu}{\sigma}}(t) = e^{-\frac{\mu t}{\sigma}}\left[e^{\frac{1}{2}\left(\frac{t}{\sigma}\right)^2}\right]$

b) IF $Y = \sum a_i X_i$ $\ni$ ALL X_i's ARE INDEPENDENT

THEN $\mu_Y = \sum a_i \mu_i$; $\sigma_Y^2 = \sum a_i^2 \sigma_i^2$

III) CHEBYCHEV'S INEQUALITY

$P[|x - \mu_x| \geq c] \leq \sigma_x^2 / c^2$

$P[|x - \mu_x| < c\sigma_x] \geq 1 - \frac{1}{c^2}$

IV) ORDERED STATISTICS

$X_{(1)}, X_{(2)} \ldots X_{(n)}$

SMALLEST BIGGEST

TEST 2 PLUG SHEET #2

I) GAMMA FUNCTION: $\Gamma(\alpha) = \int_0^\infty Y^{\alpha-1} e^{-Y} dY \quad \forall \alpha > 0$

A) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

B) $\Gamma(\alpha) = (\alpha-1)!$ FOR α A POSITIVE INTEGER

C) $\Gamma(1/2) = \sqrt{\pi}$

II) GAMMA RANDOM VARIABLE

A) $f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$ FOR $x > 0$

B) $M_X(t) = (1-\beta t)^{-\alpha}$

C) $\mu_X = \alpha\beta$; $\sigma_x^2 = \alpha\beta^2$

III) CHI-SQUARED RANDOM VARIABLE

A) IF X IS A Γ R.V. WITH $\alpha = \nu/2$ AND $\beta = 2$, THEN X IS A χ^2_ν R.V. WITH ν DEGREES OF FREEDOM

$\mu_x = \nu$; $\sigma_x^2 = 2\nu$

$M_X(t) = (1-2t)^{-\nu/2}$

B) IF $Z \sim N(0,1)$ R.V., THEN Z^2 IS A χ^2_1 R.V. (PROVE IT!)[(a)]

C) IF $X_1, X_2, \ldots, X_n$ ARE χ^2 R.V. WITH $\nu_1, \nu_2 \ldots \nu_n$ D.O.F., THEN $Y = \sum_{i=1}^{n} X_i$ IS A χ^2 R.V. WITH $\nu = \sum_{i=1}^{n} \nu_i$ D.O.F.

D) IF n SAMPLES ARE TAKEN FROM A $N(\mu, \sigma^2)$ POP, THEN $\frac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} R.V. (PROOVE ITT!)[(b)]

IV) ORDERED STATISTICS DISTRIBUTIONS

A) $F_{X(n)}(t) = [F_X(t)]^n$; $f_{X(n)}(t) = n[F_X(t)]^n f_X(t)$

B) $F_{X(1)}(t) = 1 - [1 - F_X(t)]^n$

$f_{X(1)}(t) = n[1 - F_X(t)]^{n-1} f_X(t)$ (PROVE IT!)[(c)]

V) MAXIMUM LIKELIHOOD

A) $L(\lambda) = \prod_{i=1}^{n} f(x;\lambda)$

B) MAXIMUMIZE $L(\lambda)$ TO DETERMINE M.L. ESTIMATER OF λ

VI) PROPERTIES OF ESTIMATERS

Γ IS AN ESTIMATER OF γ

A) Γ IS <u>UNBIASED</u> ESTIMATER OF γ IF $E[\Gamma] = \gamma$

($\bar{X}$ IS UNBIASED EST. OF μ; ~~σ~~ S^2 OF σ^2]

B) Γ_1 IS MORE <u>EFFICIENT</u> THAN Γ_2 IF $Var(\Gamma_1) < Var(\Gamma_2)$

C) Γ_1 IS <u>CONSISTANT</u> IF $\lim_{n\to\infty} P[|\Gamma - \gamma| > \epsilon] = 0 \ \forall \epsilon > 0$

OR $\lim_{n\to\infty} E[\Gamma] = \gamma$ AND $\lim_{n\to\infty} Var(\Gamma) = 0$

VII) CRAMER - RAO INEQUALITY (WITH CENTRAL LIMIT THEM.)

A) $VAR(\Gamma) \geq \frac{1}{n}\{E[(\frac{\delta}{\delta\gamma} \ln f(x;\gamma))^2]\}^{-1}$

B) FOR A LARGE n, Γ IS APPROX. A $N(\gamma, \sigma^2)$ R.V., WITH $\sigma^2 = [n E[\{\frac{\delta}{\delta\gamma} \ln f(x;\gamma)\}^2]]^{-1}$

PROVE ITS

a) IF X IS A $N(0,1)$ R.V., THEN X^2 IS A χ^2_1 R.V.

PROOF:

$$F_X(t) = P[X \le t]$$
$$= P[Z^2 \le t]$$
$$= P[-\sqrt{t} \le Z \le \sqrt{t}]$$
$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$
$$= 2\int_0^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

$$f_X(t) = \frac{d}{dt} F_X(t)$$
$$= \frac{d}{dt}\left[\int_0^{g(t)} h(t)\, dt\right]$$
$$= h[g(t)] \frac{d\,g(t)}{dt}$$
$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}t} \frac{d}{dt}(\sqrt{t})$$
$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}t} \left(\frac{1}{2} t^{-1/2}\right)$$
$$= \frac{1}{\sqrt{2\pi}} t^{\frac{1}{2}-1} e^{-\frac{1}{2}t}$$
$$= \frac{1}{2^{1/2}\Gamma(1/2)} t^{\frac{1}{2}-1} e^{-t/2}$$
$$= \chi^2_1$$

b) IF n SAMPLES ARE TAKEN FROM A $N(\mu, \sigma^2)$ POP., THEN $\frac{(n-1)S^2}{\sigma^2}$ IS A χ^2_{n-1} R.V.

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} = \sum_{i=1}^{n} \frac{[(X_i - \bar{X}) + (\bar{X} - \mu)]^2}{\sigma^2}$$

$$= \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{n} (\bar{X} - \mu)^2 + \overbrace{\cancel{\sum_{i=1}^{n} 2(\bar{X} - \mu)(X_i - \bar{X})}}^{0}}{\sigma^2}$$

$$= \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} + \frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

$$= \frac{(n-1)S^2}{\sigma^2} + \left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right]^2$$

$$\left[\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right]^2 = Z^2 \sim \chi^2_1$$

$$\frac{(X_i - \mu)^2}{\sigma^2} = Z^2 \sim \chi^2_1 \Rightarrow \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

c) i) $F_{X_{(n)}}(t) = [F_X(t)]^n$ AND $f_X(t) = n[F_X(t)]^{n-1} f_X(t)$

$F_{X_{(n)}}(t) = P[X_{(n)} < t]$

$= P[X_{(n)} < t \text{ AND } X_{(n-1)} < t \text{ AND} \ldots \text{ AND } X_1 < t]$

$= P[X_n < t]\, P[X_{(n-1)} < t] \ldots P[X_{(1)} < t]$

$= [F_X(t)]^n$

$f_{X_{(n)}}(t) = \frac{d}{dt}[F_{X_{(n)}}(t)]$

$= \frac{d}{dt}[F_X(t)]^n$

$= n[F_X(t)]^{n-1} \frac{d}{dt} F_X(t)$

$= n[F_X(t)]^{n-1} f_X(t)$

ii) $F_{X_{(1)}}(t) = 1 - [1 - F_X(t)]^n$; $f_{X_{(n)}}(t) = n[1 - F_X(t)]^{n-1} f_X(t)$

$F_{X_{(1)}}(t) = P[X_{(1)} < t]$

$= 1 - P[X_{(1)} > t]$

$= 1 - P[X_{(1)} > t \text{ AND } X_{(2)} > t \text{ AND} \ldots \text{ AND } X_{(n)} > t]$

$= 1 - P[X_{(1)} > t]\, P[X_{(2)} > t] \ldots P[X_{(n)} > t]$

$= 1 - [(1 - P\{X_{(1)} < t\})(1 - P\{X_{(2)} < t\}) \ldots (1 - P\{X_{(n)} < t\})]$

$= 1 - [1 - F_X(t)]^n$

$f_{X_{(1)}}(t) = \frac{d}{dt} F_{X_{(1)}}(t)$

$= \frac{d}{dt}[1 - \{1 - F_X(t)\}^n]$

$= -n[1 - F_X(t)]^{n-1} \frac{d}{dt}[-F_X(t)]$

$= n[1 - F_X(t)]^{n-1} f_X(t)$

d) i) $\underline{E(s^2) = \sigma_x^2}$

$$E(s^2) = E\left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}\right]$$

$$= E\left[\frac{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}{n-1}\right]$$

$$= \frac{E\left[\sum_{i=1}^{n} x_i^2\right] - E\left[n\bar{x}^2\right]}{n-1}$$

$$= \frac{\sum_{i=1}^{n} E\left[x_i^2\right] - n\,E\left[\bar{x}^2\right]}{n-1}$$

BUT $E[\bar{x}^2] = \sigma_{\bar{x}}^2 + \mu_{\bar{x}}^2 = \sigma_x^2/n + \mu_x^2$

AND $\sigma_x^2 = E[x_i^2] - \mu_x^2 \Rightarrow E[x_i]^2 = \sigma_x^2 + \mu_x$

$$\Rightarrow E(s^2) = \frac{\sum_{i=1}^{n}(\sigma_x^2 + \mu_x^2) - n\left[\frac{\sigma^2}{n} - \mu^2\right]}{n-1}$$

$$= \frac{(n-1)\sigma_x^2 + n\mu_x^2 - n\mu_x^2}{n-1} = \sigma_x^2$$

ii) $Var(s^2) = \frac{2\sigma^4}{n-1}$

WE KNOW FROM 'PROOVE ITT' (b) THAT

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\Rightarrow Var\left[\frac{(n-1)s^2}{\sigma^2}\right] = 2(n-1)$$

$$= \frac{(n-1)^2}{\sigma^4} Var(s^2) = 2(n-1)$$

$$\therefore Var(s^2) = \frac{2\sigma^4}{n-1}$$

TEST # 3: PLUGS

I) CONFIDENCE INTERVALS

A) STUDENT'S T DISTRIBUTION

$$T_r \triangleq Z / \sqrt{\chi^2_r / r}$$

a) $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T_{n-1}$

b) FOR LARGE n, $T_n \sim N$

B) IDENTITY DEFINITIONS

1) a) $P[-Z_{\alpha/2} \le Z \le Z_{\alpha/2}] = 1 - \alpha$

b) $P[-T_{r:\alpha/2} \le T_r \le T_{r:\alpha/2}] = 1 - \alpha$

c) $P[\chi^2_{r;1-\alpha/2} \le \chi^2_r \le \chi^2_{r:\alpha/2}] = 1 - \alpha$

2) a) $P[Z \ge Z_\alpha] = \alpha$

b) $P[\chi^2_r \ge \chi^2_{r;\alpha}] = \alpha$

c) $P[t_r \ge t_{r:\alpha}] = \alpha$

C) CONFIDENCE INTERVALS

A) μ: $\bar{X} \pm Z_{\alpha/2}\, \sigma/\sqrt{n}$; σ KNOWN [$N(\mu,\sigma^2)$

$\bar{X} \pm t_{n-1;\alpha/2}\, S/\sqrt{n}$; σ UNKNOWN "

B) σ^2: $\left[\frac{(n-1)S^2}{\chi^2_{n-1;\alpha/2}}, \frac{(n-1)S^2}{\chi^2_{n-1;1-\alpha/2}}\right]$ "

c) P: $\hat{p} \pm Z_{\alpha/2} \frac{1}{\sqrt{n}} \sqrt{\hat{p}(1-\hat{p})}$ (n LARGE)

FROM BINOMIAL POP.

d) λ: ~~UNIFORM~~ EXPODENTIAL R.V. : $f_X(x) = \lambda e^{-\lambda}$ $\forall$ $x > 0$

$$\left[\frac{1}{2n\bar{x}} \chi^2_{2n;1-\alpha/2}, \chi^2_{2n;\alpha/2} \frac{1}{2n\bar{x}}\right]$$

NOTE: $2\lambda n \bar{x} \sim \chi^2_{2n}$

II) TEST OF HYPOTHESIS

A) ERROR

1) P[TYPE I ERROR] = α
   = P[REJECTING H_0 GIVEN H_0 IS TRUE]

2) P[TYPE II ERROR] = β
   = P[ACCEPTING H_0 GIVEN H_0 AIN'T TRUE]

C) OPERATING CHARACTERISTIC CURVE

1) $C(\theta)$ = P[ACCEPTING H_0 GIVEN θ]

2) $Q(\theta) = 1 - C(\theta)$ = PWR. FUNCTION
   = P[REJECTING H_0 GIVEN θ]

D) NEYMAN PEARSON LEMMA

1) $H_0: \theta = \theta_0$; $H_1\ \theta = \theta_1$ (SIMPLE)

a) FIND $\dfrac{L(x_1, x_2, \ldots, \theta_0)}{L(x_1, x_2, \ldots, \theta_1)} \leq K$
(EVERYTHING BUT X_i'S CAN BE "SUCKED" IN BY K)

b) REDUCE TO $F(X_i) \leq K'$ (OR $F(X_i) \geq K''$)

c) FIND P[$F(X_i) > K'$ GIVEN $\theta = \theta_0$] = α

d) SOLVE FOR K'

e) REJECT FOR $F(x^u) \leq K'$ (OR $F(X_i) \geq K''$)

2) SOME "MOST PWRFULS"

a) THE MEAN: $H_0: \mu = \mu_0$; $H_1: \mu = \mu_1$
$\mu_1 > \mu_0 \Rightarrow K = Z_\alpha\, \sigma_0/\sqrt{n} + \mu_0$
$\mu_1 < \mu_0 \Rightarrow K = -Z_\alpha\, \sigma_0/\sqrt{n} + \mu_0$

b) VARIANCE: $H_0: \sigma = \sigma_0$; $H_1: \sigma = \sigma_1$
$\sigma_1 > \sigma_0 \Rightarrow K = \sigma_0^2 \chi^2_{n:\alpha}$
$\sigma_1 < \sigma_0 \Rightarrow K = \sigma_0^2 \chi^2_{n:1-\alpha}$ $K = \Sigma$

E) LIKELIHOOD RATIO TEST

1) $H_0: \underline{\theta} \in \omega \qquad H_1: \underline{\theta} \in \overline{\omega}$

$\omega \cup \overline{\omega} = \Omega \;;\; \omega \cap \overline{\omega} = \phi$

2) $L(\hat{\Omega}) =$ MAX $L(\underline{\theta})$ WITHIN Ω

$L(\hat{\omega}) =$ MAX $L(\underline{\theta})$ WITHIN ω

3) $\dfrac{L(\hat{\omega})}{L(\hat{\Omega})} = \lambda \leq A$

AGAIN: A SUCKS UP EVERYTHING BUT x_i'S

4) REDUCE TO $F(x_i) \leq A'$ (OR $F(x_i) \geq A''$)

5) FIND $P[F(x_i) \leq A'$ GIVEN $\underset{H_0}{\cancel{\underline{\theta} = \hat{\omega}}}] = \alpha$

6) REJECT H_0 IF $F(x_i) \leq A'$ (OR $F(x_i) \geq A''$)

7) SOME RESULTS

a) $H: \mu = \mu_0 \;;\; H_1: \mu \neq \mu_0$ (σ UNKNOWN)

REJECT H_0 IF

$$\frac{\bar{x} - \mu_0}{s/\sqrt{n}} \geq t_{n-1;\alpha/2} \quad \text{OR} \quad \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \leq -t_{n-1;\alpha/2}$$

b) $H_0: \sigma^2 = \sigma_0^2 \;;\; H_1: \sigma^2 \neq \sigma_0^2$

REJECT H_0 IF

$$\Sigma(x_i - \bar{x})^2 > \sigma_0^2 \chi^2_{n-1;\alpha/2}$$

$$\text{OR} \quad < \sigma_0^2 \chi^2_{n-1;\,1-\alpha/2}$$